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On Baire one Darboux functions with Lusin's condition (N)

There are many motonotonicity theorems scattering in the literatures. Recently, A. Bruckner has systematically given a very clear and flavored discussion of many of the results in [3] (cf. also [2]), where many references for other known results are also provided. A long list of some of the results was just given in B. S. Thomson's lecture [8]. In contrast to Thomson's general result in his interesting abstract setting, we are going to present a concrete monotonicity theorem related to Baire one and Darboux functions with Lusin's condition (N). It is not immediately clear to me whether Thomson's abstract result covers the result here as a particular case.

Theorem 1. Let F be a real-valued function having the following properties on an interval:

(i) F is Baire one and Darboux;

(ii) F fulfills Lusin's condition (N);

(iii) $F'(x) \geq 0$ at almost every x at which F is derivable.

Then F is monotone non-decreasing and continuous on the interval.

Note that a function F is said to be derivable at x if the derivative $F'(x)$ exists and is finite. Recall also the following terminologies :

(1) A function F is said to be Baire one on an interval I if F is the pointwise limit of a sequence of continuous functions on I.

(2) A function F is said to be Darboux on an interval I if for any a and b in I and for any y between F(a) and F(b) there exists c between a and b such that $F(c) = y$.

(3) A function F is said to fulfill Lusin's condition (N) on a set E if $|F(S)| = 0$ whenever $S \subset E$ with $|S| = 0$.

(4) A function F is said to fulfill Banach's condition (T_2) on a set E if

$|\{y : F^{-1}(y)$ uncountable in $E\}| = 0$. For a good knowledge of these concepts, see Bruckner [2] and Saks [7].

When the function F is continuous, theorem 1 is known. See, for example, page 286 in Saks [7]. To prove the general situation when F is only Baire one and Darboux instead of continuous, we need the following two results.

Theorem A. Suppose that F is Baire one and Darboux, and fulfills Banach's condition (T_2) on an interval $[a,b]$, and denote $P = \{x : 0 \le F^1(x) \le +\infty\}$, $N = \{x : -\infty \le F^1(x) \le 0\}$. Then [$P \cup N$ is uncountable and the sets $F(P)$ and $F(N)$ are measurable, and] $|F(P)| \ge F(b) - F(a)$ provided that $F(b) > F(a)$, and $|F(N)| \ge F(a) - F(b)$ provided that $F(a) > F(b)$.

Theorem B. If F is measurable and fulfills Lusin's condition (N) on an interval, then F fulfills Banach's condition (T_2) on the interval.

Theorem B was due to Banach when F is continuous (see page 284, [7]), and the general case was proved by Ellis in [4] by applying Banach's result and the well-known Lusin theorem that every measurable function is "almost" continuous (see page 73, [7]). Theorem A was established by Bruckner in [1] (cf. also [2]), generalizing theorem (6.6) on page 280 in Saks [7]. We remark that using theorem A, Bruckner [1] has established a very general monotonicity theorem related to the BVG functions (cf. theorem 2.5 on page 181, [2]), from which many monotonicity results can be obtained. Note, however, that with the aid of theorem B, theorem 1 can be proved rather easily by directly using theorem A instead of the general monotonicity theorem. The detail will appear in [6], where examples are also given to show that theorem 1 fails to hold true when the condition of being Baire one and Darboux is replaced by that of being Baire one or Darboux. However, the condition (N) can somehow be weakened. In fact, one has the following (see [6]):

Theorem 2. Suppose that F is a real-valued function which is Baire one and Darboux, and fulfills Banach's condition (T_2) on an interval, and suppose that $|F(E)| = 0$, where $E = \{x : -\infty \leq F'(x) < 0\}$. Then F is monotone non-decreasing and continuous on the interval.

As we have seen in Thomson's list in his lecture [8], many of the known monotonicity theorems have conditions related to "generalized" derivates or derivatives. It is interesting to note that theorems 1 and 2 involve only the set of points at which the function F is (ordinarily) derivable. For example, theorem 2 essentially means that for a Baire one, Darboux and (T_2) function to be monotone non-decreasing and continuous, it suffices to have some control on the set of points at which the function has negative ordinary derivatives. However, most of the known sufficient conditions for a function to fulfill the condition (T_2) or (N) are given in terms of derivates or "generalized" derivates. For some of these, see Saks [7] and Bruckner [2]. It should be noted that clearly any one of the sufficient conditions can be used to formulate a monotonicity theorem as a corollary of theorem 2 or theorem 1.

We remark that theorem 1 has been used in [6] to establish some generalizations of the Bernoulli-L'Hospital-Ostrowski rules for indeterminated forms. Applications of both theorems 1 and 2 to the study of non-absolute integrals are expected to appear somewhere else.

It is interesting to note that we have learned at the Symposium that theorem 1 has been also obtained independently by Professor K. Garg, who has mentioned it in his lectures [5] on "A new notion of derivative".

References

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