Real Analysis Exchange Vol. 7 (1981-82)

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PRODUCTS OF DERIVATIVES AND APPROXIMATE CONTINUITY

This note contains the main points of a talk given by the first author at Real Analysis Symposium.

We introduce the following notation: R is the real line; D is the system of all finite derivatives on R; if, moreover, n is a natural number, then $D^n = \{f_1 \cdots f_n;\ f_1, \ldots, f_n \in D\}$; C_{ap} is the system of all functions approximately continuous on R.

It is well known that derivatives behave badly with respect to multiplication. Simple examples show that, for instance, the product of a derivative with a differentiable function need not be a derivative. We intend to show, roughly speaking, that operations performed on derivatives lead to derivatives only in exceptional cases. As an illustration we mention that if g is a function on R such that the composite function $g \circ f$ is in D for each $f \in D$, then g is linear. Another result pointing in the same direction is the following: Let $f \in D$, let g be a function strictly convex on an open interval containing $f(R)$ and let $g \circ f \in D$. Then $f \in C_{ap}$. Taking $g(y) = y^{-1}$ for each $y \in (0,\infty)$ we see that if $f \in D$, $f > 0$ and if also $1/f \in D$, then $f \in C_{ap}$. Using the Darboux property of derivatives we get: If $f, g \in D$ and if fg is a nonzero constant, then $f, g \in C_{ap}$. This formulation suggests various generalizations. We succeeded to prove the following theorem:

Let $f_1, \ldots, f_n \in D$, $f_j > 0$ and let $f_1 \cdots f_n \in C_{ap}$. Let $a_1, \ldots, a_n \in [0,1]$, $a_1 + \cdots + a_n \leqq 1$. Then $f_1^{a_1} \cdots f_n^{a_n} \in D \cap C_{ap}$. In particular, $f_j \in C_{ap}$ $(j = 1, \ldots, n)$.

It is not surprising that products of derivatives behave badly with respect to addition. We see this well when we investigate sums of powers of derivatives. It is natural to ask, e.g., under what conditions the sum of the squares of two derivatives is in D^2. Because the product of a bounded derivative with a bounded approximately continuous functions is a derivative, the following holds: Let $h \in D$, let $\varphi, \psi \in C_{ap}$ and let the functions h, φ and ψ be bounded. Let $f = \varphi h$, $g = \psi h$. Then $f^2 + g^2 \in D^2$; actually, it is the square of the derivative $h(\varphi^2 + \psi^2)^{1/2}$. It turns out that this example is not far from the general case. Namely: Let $f, g \in D$, $f^2 + g^2 \in D^2$. Let $h = (f^2 + g^2)^{1/2}$. Then $h \in D$. If, moreover, $h \geqq 1$, then f/h, $g/h \in C_{ap}$.

We get even somehow stronger results, when we add powers with different exponents. For instance: Let $f, g, h \in D$, $1 \leqq f^2 + g^4 + h^4 \in D^2$ and $g^4 + h^4 > 0$. Then $f, g, h \in C_{ap}$.

It is probably not easy to give a simple characterization of a function in D^2 or, in general, in D^n. However, we would like to mention some partial results connected with that problem.

Let $f_j \in D$, $f_j \geqq 0$, let A_j be a set closed in R and let g_j be its characteristic function $(j = 1,\ldots,n)$. Then $\sum_{j=1}^{n} f_j g_j \in D^2$. In particular, the characteristic function g of a closed set A is always in D^2. A more detailed analysis shows that there are $\varphi,\psi \in D$ such that $0 \leqq \varphi \leqq 2$, $0 \leqq \psi \leqq 2$ and $\varphi\psi = g$. If $\emptyset \neq A \neq R$, then 2 cannot be replaced by any smaller number.

With the help of A.M. Bruckner we have proven that every function of Baire class 1 that equals zero almost everywhere is in D^2.

Now let $L = [0,\infty)$ and let D_L be the system of all derivatives on L. The meaning of D_L^n is now obvious. For each function F on L we define

$$T(F) = \sup\{\limsup_{n\to\infty} \frac{F(y_n)-F(x_n)}{y_n - x_n} ;$$

$$0 \leqq x_n < y_n,\ y_n \to 0,\ \sup_n \frac{x_n}{y_n} < 1\} .$$

We have always $T(F) \geqq \overline{F}^+(0)$. Moreover it is easy to prove that $T(F) = F'^+(0)$, if $F'^+(0)$ exists and is finite.

For each function f on L continuous on $(0,\infty)$ such that the Lebesgue integral $I = \int_0^1 f$ exists we define $S(f) = I$, if $I = \pm\infty$, and $S(f) = T(F)$, where $F(x) = \int_0^x f$, if $I \in R$. If f satisfies our conditions and if $f \in D_L$, then, obviously, $S(f) = f(0)$.

Suppose that f is a function on L that is positive and continuous on $(0,\infty)$. For each natural number n define $q_n = (S(f^{1/n}))^n$. One of our results says: Let $n > 1$. Then $f \in D_L^n$ if and only if $f(0) \geqq q_n$.

It is easy to see that $q_2 \geqq q_3 \geqq \cdots$. If, for instance, g is a nonconstant positive continuous periodic function and if $f(x) = g(1/x)$ for $x \in (0,\infty)$, then $q_1 > q_2 > q_3 > \cdots$. This shows that $D_L^n \neq D_L^{n+1}$ for each n.

Now let $q = \lim q_n$. If $f(0) > q$, then $f \in D_L^n$ for some n; if $f(0) < q$, then there is no such n. If $S(|\ln f|^a) < \infty$ for some $a > 1$, then $q = \infty$ or $q = \exp(S(\ln f))$; in general, if $q < \infty$, we have $q = \lim(\exp(S(p \vee \ln f)))(p \to -\infty)$.

The proofs of the assertions stated in this note will be published later.

Finally, we would like to mention the following problem: Is a bounded product of two (nonnegative) derivatives the product of two (nonnegative) bounded derivatives?