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LEBESGUE AREA AND DEGIORGI PERIMETER OF THE BOUNDARY OF A TOPOLOGICAL BALL IN R^n

We discuss two measurements of the image of the boundary of the standard unit ball of R^n under homeomorphisms of the ball into R^n. One is the (n-1)-dimensional Lebesgue area of C and the other is the deGiorgi perimeter of A, where A is the image of the ball and C is the image of the boundary of the ball. The results mentioned here are consequences of a paper by Casper Goffman and the author [14]. The above paper was dedicated to Lamberto Cesari in honor of this 70th birthday. As will be evident, several of the results and concepts required in the paper are due to Lamberto Cesari. The main theorem of the present paper is the last theorem of Section 3.

Remark. After the lecture was delivered at the Real Analysis Symposium, Professor Jan Mařík directed the author to the paper [15] of J. Král from which a further reference to the paper [12] of W. H. Fleming was found. Many of the results of [14] and the present paper were first proved by Fleming in [12] (and subsequently by Král in [15]) under a more general setting. We point out that the proofs given in [14] are different from those of Fleming [12] because we are able to take advantage of the added conditions not assumed by Fleming and Král. These notes follow the lecture given at the Real Analysis Symposium with appropriate additional remarks relating to the works of Fleming and Král.

1. Notations and definitions.

With $U = \{x \in R^n \mid \|x\| \le 1\}$ and $\partial U = \{x \in R^n \mid \|x\| = 1\}$, let $f : U \to A = f[U] \subset R^n$ be a homeomorphism, $C = f[\partial U]$ and $B = A \setminus C$. It is well known that B is an open set of R^n.

For each i and $x = (x_1, \ldots, x_i, \ldots, x_n) \in R^n$, let $\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, where the i^{th} coordinate of x is deleted, and let $\pi_i : R^n \to R^{n-1}$ be defined by $\pi_i(x) = \hat{x}_i$. We use the notation $x = (x_i, \hat{x}_i)$.

Since ∂U is finitely triangulable, the Lebesgue area of a continuous map $g : \partial U \to R^n$ is defined. (See [3] and [8]). We denote it by $A_{n-1}(g)$. For each i, we have associated with g the Lebesgue area $A_{n-1}(\pi_i \circ g)$. We abbreviate $A_{n-1}(f|\partial U)$ as $A_{n-1}(C)$ and $A_{n-1}(\pi_i \circ f|\partial U)$ as $A_{n-1}(C_i)$. Clearly, when $n = 2$, $A_1(g)$ is the Jordan length of g and $A_1(\pi_i \circ g)$ is the total variation $\pi_i \circ g$.

The Lebesgue measure on R^n will be denoted by μ_n.

Let $F : R^n \to R$ be Lebesgue measurable. F is said to be of bounded variation in the sense of Cesari, BVC, if

$$\sum_{i=1}^{n} \int_{R^{n-1}} \operatorname{ess} V(F|\pi_i^{-1}(\hat{x}_i))\, d\mu_{n-1}(\hat{x}_i) < \infty ,$$

where $\operatorname{ess} V(F|\pi_i^{-1}(\hat{x}_i))$ is the total variation of $F|\pi_i^{-1}(\hat{x}_i)$ using partition points consisting only of points of approximate continuity of $F|\pi_i^{-1}(\hat{x}_i)$. For later use, we abbreviate the characteristic function of a measurable set W restricted to $\pi_i^{-1}(\hat{x}_i)$ by $W_{\hat{x}_i}$.

According to deGiorgi [4,5], a measurable subset W of R^n is said to have finite perimeter if its characteristic function has partial derivatives in the distribution sense which are measures. The total variation of the resulting vector-valued measure is called the perimeter of W and is denoted by $P_{n-1}(W)$. We quote a fundamental theorem of deGiorgi.

Theorem [4]. For a bounded measurable set W of R^n, $P_{n-1}(W) < \infty$ when and only when there is a sequence of polyhedral regions $W_k (k = 1,2,...)$ such that $\mu_{n-1}(\partial W_k) \leq M < \infty$ for all k and $\mu_n((W \setminus W_k) \cup (W_k \setminus W)) \to 0$ as $k \to \infty$.

We also quote a second characterization.

Theorem. For a measurable set W of R^n, $P_{n-1}(W) < \infty$ when and only when the characteristic function of W is BVC.

2. A theorem of Federer.

At the end of his paper [10], H. Federer gives two theorems which we will formulate into one. To state the theorem we need the following terminology. We denote by C a finitely triangulable, connected, (n-1)-dimensional manifold contained in R^n and by $i_c : C \to R^n$ the inclusion map. Let E be the bounded component of $R^n \setminus C$. Then $E \cup C$ is the closure of E in R^n which will be denoted by $\mathrm{cl}E$. The two theorems from [10] are the following.

Theorem [10]. If $\mu_n(C) = 0$ and $A_{n-1}(C) < \infty$ then $P_{n-1}(E) < \infty$.

Theorem [10]. For $n = 2$, $\ell(C) < \infty$ when and only when $P_1(E) < \infty$.

When $\mu_n(C) = 0$, we have $P_{n-1}(E) = P_{n-1}(\mathrm{cl}E)$. It is also true that, for $n=2$, $\ell(C) < \infty$ implies $\mu_2(C) = 0$. Consequently, the above two theorems take the following form.

Theorem. If $\mu_n(C) = 0$ and $A_{n-1}(C) < \infty$ then $P_{n-1}(E) + P_{n-1}(\mathrm{cl}E) < \infty$. Moreover, when $n = 2$, the converse implication is true.

The first statement of the above theorem is an immediate consequence of deGiorgi's characterization of finite perimeter stated earlier. The second statement is a consequence of the BVC characterization of finite perimeter together with the following three classical theorems of real functions.

Jordan Theorem. Let $g = (g_1,\ldots,g_n) : S^1 \to R^n$ be continuous. Then $\ell(g) \le \sum_{i=1}^{n} V(g_i)$, where $V(g_i)$ is the total variation of g_i.

Banach Indicatrix Theorem. Let $g : S^1 \to R$ be continuous. Then $V(g) = \int_R M_c(y,g)\,d\mu_1(y)$, where $M_c(y,g)$ is the cardinal of the set $g^{-1}(y)$.

Theorem. Let $g = (g_1,g_2) : S^1 \to R^2$. Then, $\ell(g) < \infty$ implies $\mu_2(g(S^1)) = 0$.

The proof follows the scheme given below.

1. $P_1(E) + P_1(\mathrm{cl}E) < \infty \Rightarrow P_1(\mathrm{cl}E) < \infty$.

2. $P_1(\mathrm{cl}E) < \infty \Rightarrow \int_R M_c(x_2,\pi_2 \circ i_c)\,d\mu_1(x_2) < \infty$.

3. $V(\pi_1 \circ i_c) + V(\pi_2 \circ i_c) < \infty \Rightarrow \ell(i_c) < \infty$.

4. $\ell(i_c) < \infty \Rightarrow \mu_2(C) = 0$.

The general converse for $n \ge 3$ was not resolved in [10].

3. Status of the converse.

Due to the Jordan-Schoenflies Curve Theorem, when $n=2$, $\mathrm{cl}E$ is a topological closed ball in R^2. This need not be the case when $n \geq 3$, as is well known. We shall discuss the converse in the setting of the sets A, B and C of Section 2 where A is a topological closed ball in R^n.

Consider the case $n=3$. There are the corresponding analogies of the Jordan Theorem and the Banach Indicatrix Theorem. Namely,

Cesari-Jordan Theorem [3]. Let S^2 be a two-sphere and $g: S^2 \to R^3$ be a continuous map. Then $A_2(g) \leq A_2(\pi_1 \circ g) + A_2(\pi_2 \circ g) + A_2(\pi_3 \circ g)$.

(The above theorem has been generalized from S^2 to any finitely triangulable, connected, oriented, two-dimensional manifold in [6], [17].)

Modified Banach Indicatrix Theorem [3] and [7,8]. Let $k \geq 2$, C be a finitely triangulable, connected, oriented k-manifold and $g: C \to R^k$ be a continuous map. Then

$$A_k(g) = \int_{R^k} M(y,g)\, d\mu_k(y) ,$$

where $M(y,g)$ is a topological multiplicity function defined by means of Čech cohomology.

Unfortunately, the well known example of Besicovitch [1] shows that $A_2(C) < \infty$ does not imply $\mu_3(C) = 0$ for some topological two-sphere C in R^3. The example of Besicovitch has the property that C is the boundary of a closed topological ball A in R^3, $A_2(C) < \infty$, $\mu_3(C) > 0$ and $P_2(B) < \infty$ $(B = A \setminus C)$, the last fact being a consequence of deGiorgi's characterization of finite perimeter. So, the scheme of section 2 cannot be used. But, actually, for $n=2$, $P_1(E) + P_1(\mathrm{cl}E) < \infty$ directly implies $\mu_2(C) = 0$. Fortunately, for $n=3$, this implication can be generalized.

<u>Theorem</u> [14]. If $P_2(E) + P_2(\mathrm{cl}E) < \infty$ then $\mu_3(C) = 0$, where C is a compact, connected, two-dimensional manifold in R^3 and E is the bounded component of $R^3 \setminus C$.

We outline the proof given in [14]. Let $1 \le i \le 3$. Then $\operatorname{ess} V(C_{\hat{x}_i}) < \infty$ for μ_2-almost every $\hat{x}_i$ because

$$\int_{R^2} \operatorname{ess} V(C_{\hat{x}_i})\, d\mu_2(\hat{x}_i) = \int_{R^2} \operatorname{ess} V((\mathrm{cl}E)_{\hat{x}_i} \setminus E_{\hat{x}_i})\, d\mu_2(\hat{x}_i)$$

$$\le \int_{R^2} \operatorname{ess} V((\mathrm{cl}E)_{\hat{x}_i})\, d\mu_2(\hat{x}_i) + \int_{R^2} \operatorname{ess} V(E_{\hat{x}_i})\, d\mu_2(\hat{x}_i) < \infty .$$

If $\operatorname{ess} V(C_{\hat{x}_i}) < \infty$ and $\mu_1(C_{\hat{x}_i}) > 0$ then

$$(*_i) \begin{cases} C_{\hat{x}_i} \text{ can be decomposed in a union of a nonempty, finite,} \\ \text{disjointed collection of nondegenerate closed intervals} \\ \text{and a set of } \mu_1\text{-measure zero.} \end{cases}$$

Suppose $\mu_3(C) > 0$. Then almost every point x of C is a point of linear density in the x_1, x_2, x_3 directions. The condition $(*_i)$ implies that, contained in C , there are line segments in these directions passing through almost every x of C . Thus one can find an uncountable number of disjoint triods in C . But Moore's Triod Theorem [16] says there are no more than a countable number in a disjointed collection in the two-dimensional manifold C . This contradiction yields $\mu_3(C) = 0$.

For Besicovitch's example, the above theorem gives $P_2(A) = \infty$.

Remark. Thanks to Professor Mařík, the author learned that the above theorem was known to Král [15] and Fleming [12]. The proof given above is different from that of [15]. A proof is not given in [12]. The following problem still remains open.

Problem. Let $n \geq 4$ and A, B and C be as in Section 1. Does $P_{n-1}(A) + P_{n-1}(B) < \infty$ imply $\mu_n(C) = 0$? More generally, suppose C is a compact, connected, (n-1)-dimensional manifold in R^n and E is the bounded component of $R^n \setminus C$. Does $P_{n-1}(E) + P_{n-1}(\mathrm{cl}E) < \infty$ imply $\mu_n(C) = 0$? The general form was asked in [12].

The indicatrix theorems deal with two multiplicity functions M_c and M , the first being the crude counting function used by Banach and the second being the topological one defined in terms of Čech cohomology. There is a third one defined in terms of the stability of the map $g : C \to R^k$ [7,8,9] , where the conditions assumed are those of the Modified Banach Indicatrix Theorem above. This multiplicity function is denoted by $M_s(y,g)$ and the stable area of g is given by $S_k(g) = \int_{R^k} M_s(y,g) d\mu_k(y)$. In general, $M_s(y,g) \leq M(y,g)$. It can happen that $S_k(g) < A_k(g)$ [7]. Suppose $g : C \to R^{k+1}$ is continuous, $\pi_k : R^{k+1} \to R^k$ is the projection onto the first k coordinates and $\phi : R^{k+1} \to R^{k+1}$ is orthogonal. Then the integral geometric averages of $A_k(\pi_k \circ \phi \circ g)$ and $S_k(\pi_k \circ \phi \circ g)$ are called, respectively, the integral geometric area $M_k(g)$ and the integral geometric stable area $S_k(g)$. In general, $S_k(g) \leq M_k(g) \leq A_k(g)$.

We now quote a famous theorem of Cesari.

Theorem [2]. Let $g : S^2 \to R^2$ be continuous. Then $M(y,g) = M_s(y,g)$ except for a countable set of y in R^2.

With the aid of the above theorem and the Cesari-Jordan Theorem, Federer proves

Theorem [9]. Let $g : S^2 \to R^3$ be continuous. Then $A_2(g) = S_2(g)$.

We state a theorem from [14].

Theorem [14]. Let $n \geq 2$ and A,B and C be as in Section 1. If either $P_{n-1}(A) < \infty$ or $P_{n-1}(B) < \infty$ then $S_{n-1}(C) < \infty$.

Consequently, we have

Theorem. Let $2 \leq n \leq 3$ and A,B and C be as in Section 1. Then $\mu_n(C) = 0$ and $A_{n-1}(C) < \infty$ when and only when $P_{n-1}(A) + P_{n-1}(B) < \infty$.

The validity of the above statement for $n \geq 4$ is still unresolved.

Remarks. The paper [12] by Fleming includes much of what appears in [14] in a more general setting, and, indeed, the results in [12] are stronger.

The additional hypotheses assumed in [14] allow proofs which are different from those of Fleming. We quote the theorem of Fleming.

Theorem [12]. Let C be a finitely triangulable, connected, (n-1)-dimensional manifold in R^n and E be the bounded component of $R^n \setminus C$. Then, if $P_{n-1}(E) < \infty$ or $P_{n-1}(\mathrm{cl}E) < \infty$, the integral geometric area of C does not exceed either of $P_{n-1}(E)$ or $P_{n-1}(\mathrm{cl}E)$.

In his proof, he proves the following which is not explicitly stated in [12].

Theorem [12]. Let C be a finitely triangulable, connected, (n-1)-dimensional manifold in R^n and $\pi : R^n \to R^{n-1}$ be the natural projection. Then $M(y,\pi(C)) = M_s(y,\pi(C))$ for every $y \in R^{n-1}$. Consequently, the integral geometric area and the integral geometric stable area of C coincide. Moreover, if $\mu_n(C) = 0$ then $S_{n-1}(C) = A_{n-1}(C)$.

An analysis of the proof yields the more general theorem below.

Theorem. Let C be a finitely triangulable, connected, oriented, (n-1)-dimensional manifold and $g : C \to R^n$ be an immersion (i.e., locally a homeomorphism). Then $M(y,\pi \circ g) = M_s(y,\pi \circ g)$ for every $y \in R^{n-1}$. Consequently, the integral geometric area and the integral geometric stable area of g coincide. Moreover, if $\mu_n(g(C)) = 0$, then $S_{n-1}(g) = A_{n-1}(g)$.

Notice that in the last theorem C need not be embeddable in R^n. The example of [7] shows the immersion hypothesis is necessary. The proof can be made using the cohomology calculations in Fleming's proof, the fact that C is a locally connected continuum and the following two lemmas.

Lemma. Let X and Y be metric spaces with X compact and $g : X \to Y$ be an immersion. If Z is a closed subset of X such that $g|Z$ is a homeomorphism into Y then there is a neighborhood W of Z such that $g|W$ is an embedding of W into Y.

Proof. Since X is compact and g is an immersion, there is $\eta > 0$ such that for $x_1,x_2 \in X$ with $0 < d_X(x_1,x_2) < \eta$ it is true that $g(x_1) \neq g(x_2)$. Also, since $g|Z$ is a homeomorphism and Z is compact there is $\gamma > 0$ such that for $z_1,z_2 \in Z$ with $d_Y(g(x_1),g(x_2)) < \gamma$ it is true that

$d_x(z_1,z_2) < \eta/4$. The uniform continuity of g gives $\delta > 0$ such that $d_x(x_1,x_2) < \delta$ implies $d_y(g(x_1),g(x_2)) < \gamma/2$. We may assume $\delta < \eta/4$. Let $W = \{w \in X \mid d_x(w,Z) \leq \delta\}$. We show g is one-to-one on the compact set W . Let $w_1,w_2 \in W$ with $g(w_1) = g(w_2)$. There are $z_1,z_2 \in Z$ such that $d_x(w_1,z_1) \leq \delta$ and $d_x(w_2,z_2) \leq \delta$. From $d_y(g(z_1),g(z_2)) \leq d_y(g(z_1),g(w_1)) + d_y(g(w_1), g(w_2)) + d_y(g(w_2),g(z_2)) < \frac{\gamma}{2} + 0 + \frac{\gamma}{2} = \gamma$, we get $d_x(z_1,z_2) < \eta/4$. So, $d_x(w_1,w_2) \leq d_x(w_1,z_1) + d_x(z_1,z_2) + d_x(z_2,w_2) \leq \delta + \eta/4 + \delta \leq 3\eta/4 < \eta$. Since $g(w_1) = g(w_2)$ we have $w_1 = w_2$.

Lemma. Let X be a compact metric space and $g : X \to R$ be an immersion. Then $g|C$ is a homeomorphism for each component C of X .

Proof. Suppose C is a nondegenerate component of X . Let $x \in C$ and W be a compact neighborhood of x such that g is a homeomorphism on W . Since $g(W)$ is a compact subset of R , the components of W are points or arcs. So, x is contained in a unique component γ of $W \cap C$ and this component is an arc. Moreover, there is $\delta > 0$ so that $d(x,\gamma') \geq \delta$ for all components γ' of $W \cap C$ with $\gamma' \neq \gamma$. That is, C is locally arcwise connected. Hence C is arcwise connected. Suppose g is not one-to-one on C . That is, there are $x_1,x_2 \in C$ such that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$. There is an arc $I \subset C$ such that $x_1,x_2 \in I$. Since g is real-valued, $g|I$ is not an immersion of I into R . This contradicts the fact that g is an immersion. Consequently $g|C$ is a homeomorphism.

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