Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 858–863

V.Špitalský and V.Toma, Faculty of Mathematics and Physics, Comenius University, Mlynská dolina, 84215 Bratislava, Slovakia, e-mail: toma@fmph.uniba.sk

A THEOREM ON SEQUENCES OF DIFFERENTIABLE FUNCTIONS

1 Introduction

Some interesting results dealing with convergence of derivatives are known. We can quote e.g. results of D. Preiss and G.Petruska and M.Laczkovich ([4], [3]) stating that each Baire two function is a pointwise limit of derivatives and each Baire one function defined on a nowhere dense compact set is a uniform limit of derivatives. These results, however, don't say anything about convergence of primitives. Except the well known theorem that under uniform convergence of derivatives $(\lim f_n)' = \lim f'_n$ and some of its localizations, the literature contains few other theorems describing the relationship between f' and g, where $f = \lim f_n$, $g = \lim f'_n$. Here we try to fill in this gap for continuous derivatives by showing that the only thing we can say is $f'(x) = g(x)$ almost everywhere on a dense open set. We also show that this assertion holds in the more general case where derivatives of higher orders are considered. As a consequence we get a result related to the aforementioned theorems, namely: for every $p+1$ functions from the first Baire class defined on a nowhere dense closed set there exists a sequence of p-times continuously differentiable functions, such that the sequences of the successive derivatives converge to the corresponding function.

2 Statement of Results

The main result of this paper is the following:

Key Words: Baire 1 functions, C^p functions, pointwise limits
Mathematical Reviews subject classification: Primary: 26A24 Secondary: 26A21
Received by the editors January 9, 1995

Theorem 1 *Let $p \in \mathbb{N}$ and $g_0, g_1, \ldots, g_p : \mathbb{R} \to \mathbb{R}$. Then there is a sequence $(f_n)_{n \geq 1}$ of p-times continuously differentiable functions such that*

$$f_n \to g_0,\ f_n' \to g_1, \ldots,\ f_n^{(p)} \to g_p,$$

if and only if every g_i is Baire one,

(1) $\quad g_0^{(p)}(x) = g_1^{(p-1)}(x) = \cdots = g_p(x) \qquad$ *a.e. on a dense open set U*

and $g_0, \ldots, g_{p-1}$ are locally absolutely continuous on U.

An immediate consequence of Theorem 1 is the following

Corollary 1 *Let $p \in \mathbb{N}$ and let $F \subset \mathbb{R}$ be nowhere dense and closed. Then for every set of Baire one functions $g_0, g_1, \ldots, g_p$ defined on F there is a sequence $(f_n)_{n \geq 1}$ of p-times continuously differentiable functions defined on $\mathbb{R}$ such that*

$$f_n \to g_0,\ f_n' \to g_1, \ldots,\ f_n^{(p)} \to g_p \text{ on } F.$$

Note that Theorem 1 does not hold without the assumption that the p^{th} derivatives, $f_n^{(p)}$, are continuous. Moreover, if we only assume that the f_n are p-times differentiable, the function $g_p = \lim_{n\to\infty} f_n^{(p)}$ can be any Baire two function independent of the $g_0, \ldots g_{p-1}$. This fact easily follows from [3, Corollary 4.12] and from the fact that every Baire two function is the limit of bounded approximately continuous functions, (see [5] and [6]).

In the proof of Theorem 1 we shall use the following notation: $\mathcal{B}_1$, L^1_{loc}, C^p denotes the class of all Baire one, locally Lebesgue integrable, p-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$, respectively. Define

$$\mathcal{T}_p = \{(g_0, g_1, \ldots, g_p)\ ; \exists (f_n)_{n\geq 1} \subset C^p : f_n^{(r)} \to g_r \text{ for every } r = 0, 1, \ldots, p\},$$
$$\mathcal{T} = \bigcup_{p=1}^{\infty} \mathcal{T}_p$$

and for arbitrary $g : \mathbb{R} \to \mathbb{R}$, $M \subset \mathbb{R}$ define the function g_M by $g_M(x) = g(x)$ if $x \in M$, $g_M(x) = 0$ otherwise.

3 Proofs

The proof of the necessity is rather straightforward. Obviously we can suppose $p = 1$. For an arbitrary open interval I put $H_k = \{x \in I\ ;\ \exists n \in \mathbb{N} : |f_n'(x)| > k\}$. Since every H_k is open and $(H_k)_{k\geq 1}$ has empty intersection, by the Baire

Category Theorem there is $k \in \mathbb{N}$ and $[a,b] \subset I$ such that $H_k \cap [a,b] = \emptyset$. So $|f_n'(x)| \leq k$ on $[a,b]$ for every $n \in \mathbb{N}$ and by the Lebesgue Dominated Convergence Theorem we have for each $x \in [a,b]$

$$g_0(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}[f_n(a) + \int_a^x f_n'(t)dt] = g_0(a) + \int_a^x g_1(t)dt.$$

From this the necessity follows easily.

Lemma 1 *Let $g \in \mathcal{B}_1 \cap L^1_{loc}$; put $G^0 = g$ and let G^r be an indefinite (Lebesgue) integral of G^{r-1} $(r = 1,\ldots,p)$. Then $(G^p,\ldots,G^1,G^0 = g) \in \mathcal{T}$.*

PROOF. First suppose $g \geq 0$. Since $g \in \mathcal{B}_1$, there is a sequence of continuously differentiable functions $(g_n)_{n\geq 1}$ such that $g_n \to g$. Take arbitrary compact interval $I = [a,b]$. There is a lower semicontinuous integrable function $\psi \geq g$ on I (see e.g. [1], p.192). From [2], p.448, there is $(h_n)_{n\geq 1} \subset C(I)$ such that $0 \leq h_n \nearrow \psi$. Put

$$H_n^0(x) = \min\{g_n(x), h_n(x)\}, \quad H_n^r(x) = G^r(a) + \int_a^x H_n^{r-1}(t)dt$$

for $x \in [a,b]$, $r = 1,\ldots,p$, and define $f_n = H_n^p$. Obviously $f_n^{(p)} = H_n^0 \to G^0 = g$, $f_n^{(p-1)} = H_n^1 \to G^1$ (since $(H_n)_{n\geq 1}$ is integrably dominated by ψ), $f_n^{(p-2)} = H_n^2 \to G^2$ (since $(H_n^1)_{n\geq 1}$ is dominated by $\Psi^1(x) = G^1(a) + \int_a^x \psi(t)dt$), etc. So $(G^p,\ldots,G^0 = g) \in \mathcal{T}$ on every compact interval I, hence, it is easy to see that $(G^p,\ldots,G^0 = g) \in \mathcal{T}$ on $\mathbb{R}$.

In the general case put $g_+ = \max(g,0)$, $g_- = -\min(g,0)$. Clearly, g_+, g_- are nonnegative locally integrable Baire one functions, so from above, $(G_+^p,\ldots,G_+^0 = g_+)$, and $(G_-^p,\ldots,G_-^0 = g_-)$ belong to $\mathcal{T}$. Now the assertion follows from $G^r = G_+^r - G_-^r$ $(r = 0,\ldots,p)$. □

Lemma 2 *If $g \in \mathcal{B}_1$ and $g'(x) = 0$ on a dense open set, then, $(g,0,\ldots,0) \in \mathcal{T}$. If moreover g is locally integrable, then also $(G^p,\ldots,G^0 = g,0,\ldots,0) \in \mathcal{T}$, where $G^0,\ldots,G^p$ are such as in Lemma 1.*

PROOF. Let $g'(x) = 0$ on a dense open set U and let $\{(a_k,b_k)\}_k$ denote the system of components of U. Note that $g(x) \equiv c_k = const$ on every (a_k,b_k). Since g is a Baire one function, there is a sequence of continuous functions $(g_n)_{n\geq 1}$ converging to g; without loss of generality we can suppose that each g_n is uniformly continuous. So for $\varepsilon = \frac{1}{n}$ there is $l \in \mathbb{N}$ such that

$$|g_n(x) - g_n(y)| < \frac{1}{n} \qquad \text{whenever } |x-y| < \frac{1}{l}. \tag{2}$$

Clearly we can take a sequence $\ldots < x_{-1} < x_0 < x_1 < \ldots$, $x_i \to \infty$, $x_{-i} \to -\infty$ such that for every $k \le n$ there is $i = i_k : x_{i-1} = a_k$, $x_i = b_k$ and for every $i \ne i_1, \ldots, i_n : x_i - x_{i-1} < \frac{1}{l}$.

Put $f_n(x_i) = g_n(x_i)$ for every i and on (x_{i-1}, x_i) define f_n as follows: If $i = i_k$ for some $k \le n$, then put $f_n(x) = c_k$ for $x \in [a_k + \frac{b_k - a_k}{2n}, b_k - \frac{b_k - a_k}{2n}]$; on $(a_k, a_k + \frac{b_k - a_k}{2n}) \cup (b_k - \frac{b_k - a_k}{2n}, b_k)$ define f_n such to be p-times continuously differentiable on $[a_k, b_k]$ and $f_n^{(r)}(a_k) = f_n^{(r)}(b_k) = 0$ for every $r = 1, \ldots, p$. If $i \ne i_k$ for every $k \le n$ then (since U is dense) there is $k > n$ such that at least one of the following assertions holds:

$$\text{(i) } x_{i-1} < b_k < x_i, \quad \text{(ii) } x_{i-1} < a_k < x_i, \quad \text{(iii) } (x_{i-1}, x_i) \subset (a_k, b_k).$$

In the case (i) put $\delta = \frac{1}{2}\min\{b_k - x_{i-1}, b_k - a_k\}$ and define $f_n(x) = g_n(x_{i-1})$ for $x \in (x_{i-1}, b_k - \delta]$, $f_n(x) = g_n(x_i)$ for $x \in [b_k, x_i)$. On $(b_k - \delta, b_k) \subset (a_k, b_k)$ define f_n such that

$$\text{(3)} \qquad f_n \in C^p, \qquad f_n^{(r)}(x_{i-1}) = f_n^{(r)}(x_i) = 0 \quad \text{for } r = 1, \ldots, p;$$
$$\min\{g_n(x_{i-1}), g_n(x_i)\} \le f_n(x) \le \max\{g_n(x_{i-1}), g_n(x_i)\} \quad \text{on } [x_{i-1}, x_i].$$

Analogously define f_n in the case (ii). If (iii) is satisfied then on (x_{i-1}, x_i) we can define f_n arbitrary, requiring only that (3) holds.

The fact $f_n^{(r)} \to 0$ follows from the observation

$$\{x : f_n^{(r)}(x) \ne 0\} \subset \bigcup_{k=1}^{n} [(a_k, a_k + \tfrac{b_k - a_k}{2n}) \cup (b_k - \tfrac{b_k - a_k}{2n}, b_k)] \cup \bigcup_{k=n+1}^{\infty} (a_k, b_k).$$

The assertion $f_n(x) \to g(x)$ for $x \in U$ is obvious. Now it suffices to realize that by (2), (3) $|f_n(x) - g_n(x)| < \frac{1}{n}$ whenever $x \in \mathbb{R} - U$.

The proof of the second statement is similar to the proof of Lemma 1, only instead of $\min(g_n, h_n)$ we have to take functions f_n constructed as above such that $f_n \in C^s$, $f_n^{(r)} \to 0$ for $r = 1, \ldots, s$ and $|f_n(x) - \min\{g_n(x), h_n(x)\}| < \frac{1}{n}$ if $x \notin U$, $|f_n(x) - g(x)| < \frac{1}{n}$ if $x \in U$. (Obviously these functions also have an integrable major function on I, e.g. $\psi + g + 1$.) □

Lemma 3 *If $g \in \mathcal{B}_1$ and $g(x) = 0$ on a dense open set, then*

$$(0, \ldots, 0, g, 0, \ldots, 0) \in \mathcal{T}.$$

PROOF. Lemma 3 follows easily from Lemma 2 in the case that g is locally integrable since local integrability implies that $(G^p, \ldots, G^1, G^0 = g, 0, \ldots, 0) \in \mathcal{T}$ and $(G^p, \ldots, G^1, 0, 0, \ldots, 0) \in \mathcal{T}$, hence $(0, \ldots, 0, g, 0, \ldots, 0) \in \mathcal{T}$.

Now let g be arbitrary Baire one function such that $g(x) = 0$ on a dense open set U. Put $K = \mathbb{R} - U$; clearly K is closed and nowhere dense. Define

$K_0 = K$; if we have already defined K_β for every $\beta < \alpha$ such that $K_\beta \supset K_{\beta'}$ whenever $\beta < \beta' < \alpha$, let us define

$$M = \bigcap_{\beta<\alpha} K_\beta, \quad K_\alpha = M - \{x \in M : g_M \text{ is locally bounded at } x\}.$$

Obviously $K_\alpha \subset K_\beta$ for $\beta < \alpha$ and since g_M is Baire one, K_α is a nowhere dense closed subset of M. Therefore the sets K_α must be empty from a certain ordinal on; let α_g be the smallest ordinal α such that $K_\alpha = \emptyset$.

Now we show that $(0,\dots,0,g,0,\dots,0) \in \mathcal{T}$ by means of transfinite induction with respect to α_g. This is true if $\alpha_g = 0$, since then $g \equiv 0$ is integrable. Assume that the assertion holds for each $\alpha_g < \alpha$ and suppose that $\alpha_g = \alpha$. Put $F = \bigcap_{\beta<\alpha} K_\beta \neq \emptyset$. Obviously g_F is a Baire one function locally bounded on $\mathbb{R}$, so we have $(0,\dots,0,g_F,0,\dots,0) \in \mathcal{T}$.

If $K' \subset K - F$ is closed, then easily $K'_\beta = (K')_\beta \subset K_\beta \cap K'$ for each ordinal β, hence

$$\bigcap_{\beta<\alpha} K'_\beta \subset \bigcap_{\beta<\alpha} K_\beta \cap K' = F \cap K' = \emptyset,$$

which implies $K'_\beta = \emptyset$ for a $\beta < \alpha$. So $\alpha_{g_{K'}} < \alpha$ and $(0,\dots,0,g_{K'},0,\dots,0) \in \mathcal{T}$ by the induction hypothesis.

Let $(c_j,d_j)_{j\geq 0}$ be the system of intervals contiguous to F. Obviously for each $j \geq 0$ there is a sequence $\dots < x^j_{-1} < x^j_0 < x^j_1 < \dots$ such that $x^j_i \underset{i}{\rightarrow} d_j$, $x^j_{-i} \underset{i}{\rightarrow} c_j$. Put $M_k = K \cap \bigcup_{j=0}^{k-1} [x^j_{-(k-j)}, x^j_{k-j}]$ for every $k \geq 1$. The sets M_k are closed subsets of $K - F$, hence $(0,\dots,0,g_{M_k},0,\dots,0) \in \mathcal{T}$, i.e. there is $(h^k_n)_{n\geq 1} \subset C^p : (h^k_n)^{(r)} \underset{n}{\rightarrow} 0$ for every $r = 0,\dots,p$, $r \neq r_0$ and $(h^k_n)^{(r_0)} \underset{n}{\rightarrow} g_{M_k}$.

Define $f_n(x) = 0$ if $x \notin M_{n+1}$ and for $k = 1,2,\dots,n$ define $f_n(x) = h^k_n(x)$ if $x \in [x^j_{-(k-j)} + \delta_n, x^j_{-(k-j)+1} - \delta_n] \cup [x^j_{k-j-1} + \delta_n, x^j_{k-j} - \delta_n]$ $(j = 0,\dots,k-1)$, where $\delta_n = \frac{1}{2n}\min\{x^j_i - x^j_{i-1} : |i| + j \leq n\}$. For the others x define $f_n(x)$ such to $f_n \in C^p$ and $f^{(r)}_n(x^j_i) = 0$ for $r \neq r_0$, $f^{(r_0)}_n(x^j_i) = g(x^j_i)$ whenever $|i| + j \leq n$. Obviously $f^{(r)}_n \rightarrow 0$ for $r \neq r_0$, $f^{(r_0)}_n \rightarrow g_{K-F}$, hence $(0,\dots,0,g_{K-F},0,\dots,0) \in \mathcal{T}$. Since also $(0,\dots,0,g_F,0,\dots,0) \in \mathcal{T}$, we have $(0,\dots,0,g_{K-F} + g_F = g_K = g,0,\dots,0) \in \mathcal{T}$. □

PROOF OF SUFFICIENCY. Let $g^{(p)}_0(x) = \dots = g_p(x)$ a.e. on a dense open set U. Since $g_p \in \mathcal{B}_1$, g_p is locally bounded on a dense open set and so we can suppose that g_p (and hence each g_r) is locally bounded at every $x \in U$. Put $h_r =$

$(g_r)_U$, $k_r = (g_r)_{\mathbb{R}-U}$ for $r = 0,\dots,p$. By Lemma 3 $(0,\dots,0,k_r,0,\dots,0) \in \mathcal{T}$, so to finish the proof it suffices to show that $(h_0,\dots,h_p) \in \mathcal{T}$.

Let $\{(a_k,b_k)\}_k$ be the system of components of U and let $c_k \in (a_k,b_k)$. Obviously from the assumptions of Theorem 1 and from the fact that every g_r is locally bounded on U we have that each h_{r-1} is an indefinite integral of h_r on every (a_k,b_k). Hence by Lemma 1 there are sequences $(l_n^k)_{n\geq 1} \subset C^p$ such that $(l_n^k)^{(r)} \underset{n}{\longrightarrow} h_r$ on (a_k,b_k) for every $r = 0,\dots,p$. Now define functions l_n as follows: Put $l_n(x) = 0$ for $x \in \mathbb{R} - U$, $l_n(x) = l_n^k(x)$ for $x \in [a_k + \frac{b_k-a_k}{2n}, b_k - \frac{b_k-a_k}{2n}]$ and on $(a_k, a_k + \frac{b_k-a_k}{2n})$, $(b_k - \frac{b_k-a_k}{2n}, b_k)$ define l_n such that $l_n \in C^p$ and $l_n^{(r)}(a_k) = l_n^{(r)}(b_k) = 0$ for every $r = 1,\dots,p$, $k \in \mathbb{N}$ (existence of such functions is obvious). It is now easy to see that $f_n^{(r)} \to h_r$ for each $r = 0,\dots,p$, and hence $(h_0,\dots,h_p) \in \mathcal{T}$. □

References

[1] D. L. Cohn, *Measure Theory,* Birkhäuser, Boston (1980).

[2] I. P. Natanson, *Theory of Functions of a Real Variable,* Moscow (1957) (*Russian*).

[3] G. Petruska and M. Laczkovich, *Baire 1 functions, approximately continuous functions and derivatives,* Acta Math. Acad. Sci. Hung. **25** (1974) 189–212.

[4] D. Preiss, *Limits of derivatives and Darboux-Baire functions,* Rev. Roum. Math. Pures et Appl. **14** (1969) 1201–1206.

[5] G. Petruska and M. Laczkovich, *A theorem on approximately continuous functions*, Acta Math. Acad. Sci. Hung. **24** (1973) 383–387.

[6] D. Preiss, *Limits of approximately continuous functions*, Czech. Math. J. **21** (1971) 371–372.