Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 831–841

Russell A. Gordon, Department of Mathematics, Whitman College, Walla Walla, WA 99362 e-mail: gordon@whitman.edu

SOME COMMENTS ON AN APPROXIMATELY CONTINUOUS KHINTCHINE INTEGRAL

Abstract

The Khintchine integral is not comparable to the approximately continuous Perron integral. That is, there are functions that are integrable in one sense but not the other. There have been several attempts to define an integration process that includes both of these integrals. This paper points out an error in one of these proofs and defines another integration process that includes both of the integrals mentioned above.

Is there an integration process that includes both the Denjoy integral and the approximately continuous Perron integral? An affirmative answer to this question was given by Kubota [4]. However, a careful reading of his argument reveals a logical error. This paper represents another attempt to provide an affirmative answer to the opening question.

The focus of this paper is on integration processes that recover an approximately continuous function from its approximate derivative. Integrals with this property are often referred to as approximately continuous integrals since the indefinite integral is approximately continuous rather than continuous. The usual starting points for defining integrals of this type are the integrals that recover a continuous function from its derivative. There are three such integrals, namely the Denjoy, Perron, and Henstock integrals, and they are all equivalent. The idea is to modify the definitions of these integrals to obtain an integral that recovers an approximately continuous function from its approximate derivative. The necessary modifications for the Perron and Henstock integrals are relatively easy and result in integrals known as the approximately continuous Perron integral (AP integral) and the approximately continuous Henstock integral (AH integral), respectively. It is generally agreed that (with

Key Words: Khintchine integral, AP integral, ACG function
Mathematical Reviews subject classification: Primary: 26A39
Received by the editors December 6, 1994

the proper definitions) the AP and AH integrals are equivalent. The situation for the Denjoy integral is not as easy. There have been several attempts to define an approximately continuous Denjoy integral (see [2], [6], and [7]), but none of them is very satisfying in the sense of a natural generalization. One is left wondering if maybe the Denjoy integral is an unsuitable starting point for such a generalization.

Another integral that is relevant to this situation is the Khintchine integral. The Khintchine integral (K integral) recovers a continuous function from its approximate derivative and includes the Denjoy integral. It is known that the AP and AH integrals and the Khintchine integral are not comparable. To say that the K and AH integrals are not comparable means that there are K integrable functions that are not AH integrable and there are AH integrable functions that are not K integrable. The fact that there exist AH integrable functions that are not K integrable is easy to see; simply let F be any approximately differentiable function that is not continuous and look at F'_{ap}. It is a bit surprising that there are K integrable functions that are not AH integrable. An example of such a function can be found in [3].

In [4], Kubota attempts to define an integral that includes both the K and AP integrals. His method is to generalize the Khintchine integral. As will be pointed out below, Kubota's proof that his integral includes the AP integral contains an error. It is not at all clear that Kubota's proof can be repaired. Rather than embark on a remodeling mission, this paper presents several possible definitions for an AK integral, including Kubota's definition. We will show that there is an AK integral that includes both the K and AH integrals. The question of where Kubota's integral fits will be left unresolved, but placed in a simpler context.

Recall that a function $f : [a,b] \to R$ is Khintchine integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to R$ such that $F'_{\text{ap}} = f$ almost everywhere on $[a,b]$. To define an AK integral, the condition that F be ACG on $[a,b]$ must be weakened. We will assume that the reader is familiar with the definitions and properties of BVG and ACG functions. Many of the properties of these functions can be found in the book by Saks [11]. Another source for this information is the recent book by Gordon [3].

In order to proceed, some basic terminology and notation is needed. A portion of a set E is a nonempty set of the form $E \cap I$ where I is an open interval. In the sequel, we will be looking for certain behavior on portions of closed sets. Since this will trivially be the case at isolated points, it is sufficient to consider nonempty perfect sets. Let P be a nonempty perfect set. It will often be convenient to consider perfect portions of P. A perfect portion of P is a set of the form $P \cap [c,d]$ where $P \cap (c,d) \neq \emptyset$, $c, d \in P$, and $P \cap [c,d]$ is a perfect set. It is always possible to convert a portion of P to a perfect portion of P by shrinking the interval. This terminology, by the way, is not standard.

Let $F : [a,b] \to R$ and let $E \subseteq [a,b]$. The symbol $F|_E$ represents the restriction of the function F to the set E and $\overline{E}$ represents the closure of E. The function F satisfies condition (N) on E if $\mu^*(F(A)) = 0$ for every set $A \subseteq E$ of measure zero. (Here $\mu^*(A)$ represents the outer Lebesgue measure of A.) For ease of reference, we record three theorems that will be useful later. The proofs of these theorems can be found in the books by Saks and Gordon.

Theorem A *If F is ACG on $[a,b]$, then $[a,b] = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed and F is AC on each E_n.*

Theorem B *Let $F : [a,b] \to R$, let $E \subseteq [a,b]$ be closed, and suppose that $F|_E$ is continuous on E. Then F is ACG on E if and only if every nonempty perfect subset of E contains a portion on which F is AC.*

Theorem C *Let $F : [a,b] \to R$, let $E \subseteq [a,b]$ be closed, and suppose that F is BVG on E and $F|_E$ is continuous on E. Then F is ACG on E if and only if F satisfies condition (N) on E.*

Here are three different candidates for a definition of an AK integral. Each one represents a modification of the ACG condition. The first definition is the one given by Kubota in [4].

Definition 1 *A function $F : [a,b] \to R$ is ACG_c on $[a,b]$ if it satisfies the following properties:*

(i) F is approximately continuous on $[a,b]$;

(ii) $[a,b] = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed and F is AC on each E_n.

A function $f : [a,b] \to R$ is AK_c integrable on $[a,b]$ if there exists an ACG_c function $F : [a,b] \to R$ such that $F'_{\rm ap} = f$ almost everywhere on $[a,b]$.

Definition 2 *A function $F : [a,b] \to R$ is ACG_p on $[a,b]$ if it satisfies the following properties:*

(i) F is approximately continuous on $[a,b]$;

(ii) $[a,b] = \bigcup_{n=1}^{\infty} E_n$ where each E_n is measurable and F is AC on each E_n;

(iii) every nonempty perfect set in $[a,b]$ contains a portion on which F is AC.

A function $f : [a,b] \to R$ is AK_p integrable on $[a,b]$ if there exists an ACG_p function $F : [a,b] \to R$ such that $F'_{\rm ap} = f$ almost everywhere on $[a,b]$.

Definition 3 *A function $F : [a,b] \to R$ is BVG_N on $[a,b]$ if it satisfies the following properties:*

(i) F is approximately continuous on $[a,b]$;

(ii) F is BVG on $[a,b]$;

(iii) F satisfies condition (N) on $[a,b]$.

A function $f : [a,b] \to R$ is AK_N integrable on $[a,b]$ if there exists a BVG_N function $F : [a,b] \to R$ such that $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$.

By the Baire Category Theorem, every function that is ACG_c on $[a,b]$ is also ACG_p on $[a,b]$. Since a function that is AC on a set is BV and satisfies condition (N) on that set, every function that is ACG_p on $[a,b]$ is also BVG_N on $[a,b]$. (In particular, in all three cases, the function F is BVG on $[a,b]$ and hence approximately differentiable almost everywhere on $[a,b]$.) Consequently, with an obvious abuse of notation,

$$K \subseteq AK_c \subseteq AK_p \subseteq AK_N.$$

Since an indefinite K integral must be continuous, the first inclusion is certainly proper. Are the other two inclusions proper?

Before answering this question, we should pause to make certain that the integrals defined above are unique except for an additive constant. In other words, if F and G are BVG_N on $[a,b]$ and if $F'_{\mathrm{ap}} = G'_{\mathrm{ap}}$ almost everywhere on $[a,b]$, does it necessarily follow that F and G differ by a constant? To resolve this issue, it is sufficient to prove the following theorem.

Theorem 1 *Suppose that $F : [a,b] \to R$ is BVG_N on $[a,b]$. If $F'_{\mathrm{ap}} \geq 0$ almost everywhere on $[a,b]$, then F is nondecreasing on $[a,b]$.*

This theorem is a simple consequence of a general monotonicity theorem proved by Bruckner. See [1] for a proof of the following result.

Monotonicity Theorem *Let $\mathcal{P}$ be a function-theoretic property that satisfies the following two conditions:*

(i) Any continuous BV function that satisfies property $\mathcal{P}$ on $[a,b]$ is nondecreasing on $[a,b]$.

(ii) Any Darboux function in Baire class one which satisfies property $\mathcal{P}$ on $[a,b]$ is BVG on $[a,b]$.

Then any Darboux function in Baire class one which satisfies property $\mathcal{P}$ on $[a, b]$ is continuous and nondecreasing on $[a, b]$.

In order to use the Monotonicity Theorem to prove Theorem 1, we make the following observations. Define $\mathcal{P}$ as follows: a function F has property $\mathcal{P}$ on $[a, b]$ if it is BVG on $[a, b]$, satisfies condition (N) on $[a, b]$, and satisfies $F'_{\text{ap}} \geq 0$ almost everywhere on $[a, b]$. Suppose that F is a continuous, BV function that satisfies property $\mathcal{P}$ on $[a, b]$. It follows that (since F satisfies condition (N) on $[a, b]$) F is AC on $[a, b]$ (see [3] or [11]). Since $F' = F'_{\text{ap}} \geq 0$ almost everywhere on $[a, b]$, the function F is nondecreasing on $[a, b]$. Hence, condition (i) holds. The second condition is obviously satisfied given the definition of $\mathcal{P}$. The theorem now follows since an approximately continuous function is a Darboux function in Baire class one.

We now return to a discussion of the relationships between the AK integrals defined above. The next theorem shows that the AK_c and AK_p integrals are equivalent. The following lemma, which originates in a paper by Romanovski [10], will be used in the proof.

Romanovski's Lemma *Let $\mathcal{F}$ be a family of open intervals in (a, b) and suppose that $\mathcal{F}$ has the following properties:*

(1) If (α, β) and (β, γ) belong to $\mathcal{F}$, then (α, γ) belongs to $\mathcal{F}$.

(2) If (α, β) belongs to $\mathcal{F}$, then every open interval in (α, β) belongs to $\mathcal{F}$.

(3) If (α, β) belongs to $\mathcal{F}$ for every interval $[\alpha, \beta] \subseteq (c, d)$, then (c, d) belongs to $\mathcal{F}$.

(4) If all of the intervals contiguous to the perfect set $E \subseteq [a, b]$ belong to $\mathcal{F}$, then there exists an interval I in $\mathcal{F}$ such that $I \cap E \neq \emptyset$.

Then $\mathcal{F}$ contains the interval (a, b).

Theorem 2 *If $F : [a, b] \to R$ is ACG_p on $[a, b]$, then F is ACG_c on $[a, b]$.*

PROOF. Let $\mathcal{F}$ be the collection of all open intervals I in (a, b) such that F is ACG_c on $\overline{I}$. We will show that $\mathcal{F}$ satisfies the four conditions of Romanovski's Lemma. It is easy to verify that $\mathcal{F}$ satisfies conditions (1) and (2). Suppose that (α, β) belongs to $\mathcal{F}$ for every interval $[\alpha, \beta] \subseteq (c, d)$. Let $\{\epsilon_n\}$ be a decreasing sequence in $(0, (d-c)/2)$ that converges to 0. For each n , there exists a sequence $\{E_n^k\}$ of closed sets such that $[c+\epsilon_n, d-\epsilon_n] = \bigcup_{k=1}^{\infty} E_n^k$ and F is AC on each E_n^k. Since

$$[c, d] = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_n^k \cup \{c\} \cup \{d\},$$

and since F is AC on each of these closed sets, it follows that (c, d) belongs to $\mathcal{F}$. This shows that $\mathcal{F}$ satisfies condition (3).

Now suppose that all of the intervals contiguous to the perfect set $E \subseteq [a, b]$ belong to $\mathcal{F}$. Since F is ACG_p on $[a, b]$, there exists a perfect portion $P = E \cap [c, d]$ of E such that F is AC on P. Let $[c, d] - P = \bigcup_{n=1}^{\infty} (c_n, d_n)$. By hypothesis, the function F is ACG_p on each $[c_n, d_n]$. For each n, let $\{E_n^k\}$ be a sequence of closed sets such that $[c_n, d_n] = \bigcup_{k=1}^{\infty} E_n^k$ and F is AC on each E_n^k. Now

$$[c, d] = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_n^k \cup P$$

and it follows that (c, d) belongs to $\mathcal{F}$. Hence, the family $\mathcal{F}$ satisfies condition (4). By Romanovski's Lemma, the interval (a, b) belongs to $\mathcal{F}$. That is, the function F is ACG_c on $[a, b]$. This completes the proof.

To examine the relationship between the AK_p and AK_N integrals, the concept of B_1^* function is useful. A function $F : [a, b] \to R$ is a B_1^* function on $[a, b]$ if every nonempty perfect set E in $[a, b]$ contains a perfect portion P such that $F|_P$ is continuous on P. See O'Malley [9] for some of the properties of B_1^* functions.

Theorem 3 *Suppose that $F : [a, b] \to R$ is BVG_N on $[a, b]$. Then F is a B_1^* function on $[a, b]$ if and only if F is ACG_p on $[a, b]$.*

PROOF. Suppose first that F is a B_1^* function on $[a, b]$. Let $E \subseteq [a, b]$ be a nonempty perfect set. Since F is a B_1^* function on $[a, b]$, there exists a perfect portion $P = E \cap [c, d]$ of E such that $F|_P$ is continuous on P. By Theorem C, the function F is ACG on P. By Theorem B, there is a portion $P \cap I$ of P such that F is AC on $P \cap I$. We may assume that $I \subseteq (c, d)$ from which it follows that $P \cap I = E \cap I$. Consequently, there exists a portion of E on which F is AC. Therefore, the function F is ACG_p on $[a, b]$.

Now suppose that F is ACG_p on $[a, b]$ and let $E \subseteq [a, b]$ be a nonempty perfect set. By hypothesis, there exists a perfect portion $P = E \cap [c, d]$ of E such that F is AC on P. Since F is AC on P, the function $F|_P$ must be continuous on P. It follows that F is a B_1^* function on $[a, b]$. This completes the proof.

So where do we stand? Focusing on the indefinite integrals and abusing the notation,

$$K \subseteq AK_c = AK_p = AK_N \cap B_1^* \subseteq AK_N.$$

In [3] (see Chapter 16 and its exercises) , it is shown that an indefinite AH integral is BVG_N on $[a, b]$. This provides an answer to the opening question in this paper. The AK_N integral includes both the K (and hence Denjoy) and AH integrals. Kubota (see [4, Theorem 2]) claims that the AK_c integral includes both the K and AP integrals. However, the proof of this theorem contains an error. The error occurs in the third paragraph of the proof and its essence is the following assertion.

- If $\{U_n\}$ is a sequence of ACG_c functions defined on $[a, b]$, then there exists a sequence $\{E_k\}$ of closed sets such that $[a, b] = \bigcup_{k=1}^{\infty} E_k$ and every U_n is AC on each E_k.

In particular (use the Baire Category Theorem), there exists some interval I such that every U_n is AC on I. For a counterexample, let $\{r_n\}$ be the sequence of rational numbers in $[a, b]$. For each n, define U_n on $[a, b]$ by

$$U_n(x) = \begin{cases} 0, & \text{if } a \leq x \leq r_n \\ \frac{1}{n}(x - r_n)^2 \sin\left(\pi/(x - r_n)^2\right), & \text{if } r_n < x \leq b. \end{cases}$$

Note that U_n is not AC on any interval that contains r_n as an interior point. The sequence $\{U_n\}$ converges uniformly to 0 on $[a, b]$, but there is no interval on which every U_n is AC.

C. M. Lee ([5]) recognized the oversight in Kubota's proof and offered a proof that a closed set decomposition existed. However, his proof contains an error as well. Assuming familiarity with this proof, the error occurs at the top of page 72 where it is stated that M_* is uCM on $[a, b]$. Let $\{(a_n, b_n)\}$ be a sequence of disjoint open intervals in $(0, 1)$ such that $\{a_n\}$ is a decreasing sequence that converges to 0 and 0 is a point of dispersion of the set $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$. Let c_n be the midpoint of $[a_n, b_n]$. Define $M : [0, 1] \to R$ by

$$M(x) = \begin{cases} 2, & \text{if } x \in [0, 1] - U \\ 1, & \text{if } x = c_n \end{cases}$$

and letting M be linear on each of the intervals $[a_n, c_n]$ and $[c_n, b_n]$. The function M is continuous on $(0, 1]$ and approximately continuous at 0. It follows that M is a Darboux function and hence uCM on $[0, 1]$. Now let E be the closed set $\{0\} \cup \{c_n : n \in Z^+\}$ and let M_* be the linear extension of $M|_E$ from E to $[0, 1]$. Then

$$M_*(x) = \begin{cases} 2, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$$

is not uCM on $[0, 1]$. Since the assumption that M_* is uCM is crucial for the rest of the argument, Lee's proof is not valid.

A more recent attempt to correct Kubota's error was made by Lin [8]. Unfortunately, this proof also contains an unjustified claim. In the proof of Lemma 3, Lin says to use $X = E_{ni}$ in Lemma 1. However, the construction of E_{ni} is not the same as that required by Lemma 1. For Lemma 1, the set X must be the set of all points $x \in [a,b]$ with a certain property. Here we are only taking points from E_n with a certain property. With $A = n$, $\lambda = 0.5$, and $\delta = 1/i$, the set X contains E_{ni} but may not equal E_{ni}.

We thus have the following two unanswered questions.

1. Is every BVG_N function a B_1^* function?

2. Is every indefinite AH (or AP) integral a B_1^* function?

If the answer to the first question is yes, then the answer to the second is yes as well. If the answer to the first question is no, it is still possible for the second question to have an affirmative answer. In this case, Kubota's result will have been shown to be correct, but with a different proof. At this time, both of these questions remain open.

We close with one final related result. The typical example of an indefinite AH integral is an approximately continuous function that is approximately differentiable nearly everywhere (except for a countable set). In this particular case, the function is also an indefinite AK_c integral. The proof of this result is a modification of an argument given by Tolstoff [12].

Theorem 4 *Let $F : [a,b] \to R$ be approximately continuous on $[a,b]$. If F is approximately differentiable nearly everywhere on $[a,b]$, then there exists a sequence $\{E_n\}$ of closed sets such that $[a,b] = \bigcup_{n=1}^{\infty} E_n$ and $F|_{E_n}$ is continuous on E_n for each n. Consequently, the function F is a B_1^* function.*

PROOF. Define sets A, B, and C as follows:

$$A = \{x \in (a,b) : F'_{\text{ap}}(x) > -1\};$$
$$B = \{x \in (a,b) : F'_{\text{ap}}(x) < 1\};$$
$$C = \{x \in (a,b) : F'_{\text{ap}}(x) \text{ does not exist}\} \cup \{a,b\}.$$

For each $x \in A$, let

$$R_x = \left\{t \in (a,b) : \frac{F(t)-F(x)}{t-x} > -1\right\}.$$

Note that x is a point of density of R_x. Consequently, for each $x \in A$ there exists $\eta_x > 0$ such that $(x-\eta_x, x+\eta_x) \subseteq (a,b)$ and

$$\frac{\mu(R_x \cap [x-h,x])}{h} > \frac{1}{2} \quad \text{and} \quad \frac{\mu(R_x \cap [x,x+h])}{h} > \frac{1}{2}$$

for all $0 < h < \eta_x$. For each positive integer n, let $A_n = \{x \in A : \eta_x > 1/n\}$. We will show that $F|_{\overline{A}_n}$ is continuous on $\overline{A}_n$ for each n.

Fix n. We first show that

$$\lim_{\substack{x \to z \\ x \in A_n}} F(x) = F(z)$$

for each $z \in \overline{A}_n$. To avoid the trivial case, suppose that z is a limit point of A_n. Suppose that

$$\lim_{\substack{x \to z \\ x \in A_n}} F(x) \neq F(z).$$

Then there exists a sequence $\{x_k\}$ in A_n such that $\{x_k\}$ converges to z, but $\{F(x_k)\}$ does not converge to $F(z)$. Of the many similar cases, we will assume that the sequence $\{x_k\}$ is decreasing and that $\lim_{k \to \infty} F(x_k) > F(z)$. (The limit in this case may be ∞.) We may also assume that there exists a positive number $\delta < 1/n$ such that $F(x_k) > F(z) + 2\delta$ for all k and $\{x_k\} \subseteq (z, z+\delta)$. For each $k \geq 2$, let $H_k = R_{x_k} \cap [x_k, x_{k-1}]$. If $t \in H_k$, then

$$\frac{F(t) - F(x_k)}{t - x_k} > -1$$

which in turn implies that

$$F(t) > F(x_k) - (t - x_k) > F(z) + 2\delta - \delta = F(z) + \delta.$$

It follows that $F(t) > F(z) + \delta$ for all $t \in \bigcup_{k=2}^{\infty} H_k$. Now for each integer $q \geq 2$,

$$\begin{aligned} \frac{1}{x_q - z}\mu\Big(\bigcup_{k=2}^{\infty} H_k \cap [z, x_q]\Big) &= \frac{1}{x_q - z} \sum_{k=q+1}^{\infty} \mu(H_k) \\ &> \frac{1}{x_q - z} \sum_{k=q+1}^{\infty} (x_{k-1} - x_k)/2 = \frac{1}{2}. \end{aligned}$$

This shows that the set $\{t \in [a,b] : |F(t) - F(z)| < \delta\}$ does not have z as a point of density. This contradicts the fact that F is approximately continuous at z. Hence,

$$\lim_{\substack{x \to z \\ x \in A_n}} F(x) = F(z)$$

for each $z \in \overline{A}_n$.

Now suppose that $z \in \overline{A}_n$ and let $\{x_k\}$ be a sequence in $\overline{A}_n$ that converges to z. We will show that the sequence $\{F(x_k)\}$ converges to $F(z)$. By the result in the previous paragraph, for each positive integer k there exists $y_k \in A_n$ such that

$$|y_k - x_k| < \frac{1}{k} \quad \text{and} \quad |F(y_k) - F(x_k)| < \frac{1}{k}.$$

Since $\{y_k\}$ is a sequence in A_n that converges to z, the result in the previous paragraph shows that $\{F(y_k)\}$ converges to $F(z)$. It follows that $\{F(x_k)\}$ converges to $F(z)$ and we conclude that $F|_{\overline{A}_n}$ is continuous at z. Since z was an arbitrary point in $\overline{A}_n$, the function $F|_{\overline{A}_n}$ is continuous on $\overline{A}_n$.

We have thus shown that $A \subseteq \bigcup_{n=1}^{\infty} \overline{A}_n$ where $F|_{\overline{A}_n}$ is continuous on $\overline{A}_n$ for each n. Similarly, it can be shown that $B \subseteq \bigcup_{n=1}^{\infty} \overline{B}_n$ where $F|_{\overline{B}_n}$ is continuous on $\overline{B}_n$ for each n. Since C is a countable set, it can be written as a countable union of single points; $C = \bigcup_{n=1}^{\infty} \{c_n\}$. Now

$$[a,b] = \bigcup_{n=1}^{\infty} \overline{A}_n \cup \bigcup_{n=1}^{\infty} \overline{B}_n \cup \bigcup_{n=1}^{\infty} \{c_n\},$$

and the restriction of F to each of these closed sets is continuous on that closed set. This completes the proof.

I would like to thank the referee for pointing out two additional references.

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lect. Notes in Math, Vol. 659, Springer-Verlag, Heidelberg, 1978.

[2] P. S. Bullen, *The Burkill approximately continuous integral*, J. Austral. Math. Soc. (Series A) **35** (1983), 236–253.

[3] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, **4**, American Mathematical Society, Providence, 1994.

[4] Y. Kubota, *An integral of the Denjoy type*, Proc. Japan Acad. **40** (1964), 713–717.

[5] C. M. Lee, *An analogue of the theorem of Hake–Alexandroff–Looman*, Fund. Math. C (1978), 69–74.

[6] P. Y. Lee, *Lanzou lectures on Henstock integration*, World Scientific, Singapore, 1989.

[7] K. Liao, *On the descriptive definition of the Burkill approximately continuous integral* , Real Analysis Exchange, **18** (1992-93), 253–260.

[8] Y. Lin, *A note on that Kubota's AD–integral is more general than Burkill's AP–integral*, J. Mathematical Study, bf 27 (1994), 116–120.

[9] R. J. O'Malley, *Baire* 1, Darboux functions*, Proc. Amer. Math. Soc. **60** (1976), 187–192.

[10] P. Romanovski, *Essai d'une exposition de l'intégrale de Denjoy sans nombres transfini*, Fund. Math. **19** (1932), 38–44.

[11] S. Saks, *Theory of the integral*, 2nd. ed. revised, Hafner, New York, 1937.

[12] G. Tolstoff, *Sur la dérivée approximative exacte*, Mat. Sb. 4 (46) (1938), 499-504.