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ON LOCAL RELATIVE CONTINUITY

Abstract

A local version of the relative continuity of J. Chew and J. Tong is introduced and observed to have applications to connected functions and separate continuity. A strong version of Levine's decomposition of continuity is also obtained.

1 Introduction

In [1] N. Levine introduced the notions of $w*$ continuity and weak continuity. In [2] the second-named author introduced the notion of local $w*$ continuity and showed that $w*$ continuity implies local $w*$ continuity, but not conversely. Here we present a strict generalization of local $w*$ continuity, which we call *local relative continuity.* Local relative continuity is a generalization of the closed graph property for functions of the form $f : X \to Y$, where X is a space and Y is locally compact and Hausdorff. Results are presented which apply the notion of local relative continuity in the contexts of the closed graph property, connected functions, separate and joint continuity, weak continuity, and related concepts, including certain decompositions of continuity.

Throughout this paper, a function on a topological space X into a topological space Y, will be represented by $f : X \to Y$, where X and Y have no particular properties unless otherwise indicated.

2 Preliminaries and Definitions

In [3] it was shown that, for a function $f : X \to Y$, where Y is a locally compact Hausdorff space, if f has the closed-graph property, then f is locally

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$w*$ continuous. A function $f : X \to Y$ is *locally $w*$ continuous* if and only if there exists an open basis $\mathbf{B}$ for the topology on Y such that for any V in $\mathbf{B}$, $f^{-1}[Fr(V)]$ is closed in X, where Fr denotes the frontier operator. If $f^{-1}[Fr(V)]$ is closed for every open $V \subseteq Y$, then f is said to be $w*$ continuous [1]. Here we further generalize the closed-graph property by showing that local $w*$ continuity implies local relative continuity, but not conversely.

In [4] a function $f : X \to Y$ is called *relatively continuous* at $x \in X$ if and only if for any open set $V \subseteq Y$, where $f(x)$ is contained in V, the set $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$, where Cl denotes the closure operator. If this condition is satisfied for each $x \in X$, then f is said to be relatively continuous. The notion of relative continuity is generalized by the following:

Definition 1 *A function $f : X \to Y$ is locally relatively continuous if there exists an open basis $\mathbf{B}$ for the topology on Y such that $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$ for any $V \in \mathbf{B}$.*

3 A Generalization of the Closed-Graph Property

Theorem 1 *If $f : X \to Y$ is locally $w*$ continuous, then f is locally relatively continuous.*

PROOF. There is an open basis $\mathbf{B}$ for the topology on Y such that for any $V \in \mathbf{B}$, $f^{-1}[Fr(V)]$ is closed in X. Let V be any set in $\mathbf{B}$. Then $V = Cl(V) - Fr(V)$, and $f^{-1}(V) = f^{-1}[Cl(V)] - f^{-1}[Fr(V)] = f^{-1}[Cl(V)] \cap (X - f^{-1}[Fr(V)])$. Since f is locally $w*$ continuous, the set $X - f^{-1}[Fr(V)]$ is open in X. Hence $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$. □

Remark 1 We first discovered that relative continuity as formulated in [4] is equivalent to the $w*$ continuity of [1]. It is interesting that local relative continuity is strictly weaker than local $w*$ continuity. For example, consider the function $f : [0, 1] \to [-2, 2]$ defined by

$$f(x) = \begin{cases} \frac{3}{2} & \text{if } x = 0 \\ \sin \frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

If we choose a basis consisting of open intervals of length at most $\frac{1}{4}$ relativized to $[-2, 2]$, it is readily seen that f is locally relatively continuous. However, f is not locally $w*$ continuous for any open basis. For consider the open interval $(-\frac{1}{2}, \frac{1}{2})$ in the range of f, and let y be a point in $(-\frac{1}{2}, \frac{1}{2})$. Then there exists an open basis element V, contained in $(-\frac{1}{2}, \frac{1}{2})$, such that $y \in V$. Since V is open, it is a union of open intervals. Let α be the infimum of the left endpoints of these intervals and let β be the supremum of the right endpoints of these intervals. Then α and β belong to the set $Fr(V)$, and we see that $f^{-1}[Fr(V)]$

has zero as a limit point, but zero is not an element of $f^{-1}[Fr(V)]$. Thus, local $w*$ continuity implies local relative continuity, but not conversely.

Corollary 1 *Let $f : X \to Y$ be a function, where Y is a locally compact Hausdorff space. If f has the closed graph property, then f is locally relatively continuous.*

PROOF. It was shown in [3] that every function with closed graph into a locally compact Hausdorff space is locally $w*$ continuous. □

Remark 2 In Remark 3 below we demonstrate that the condition of local compactness on Y is essential in Corollary 1.

4 Local Relative Continuity and Connected Functions

The next result generalizes Theorem 6 of [3]. We recall that a function $f : X \to Y$ is said to be *connected* if and only if the image of each connected set in X is a connected set in Y.

Theorem 2 *Let X be a locally connected space and let $f : X \to Y$ be a connected function. If f is locally relatively continuous, f is continuous.*

PROOF. Let **B** be an open basis for the topology on Y such that for any $V \in \mathbf{B}$, $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$. It is sufficient to show that for any $V \in \mathbf{B}$, $f^{-1}(V)$ is open in X. Assume there exists $V \in \mathbf{B}$ such that $f^{-1}(V)$ is not open in X. Then there is a point $x' \in f^{-1}(V)$ such that any open set in X containing x' meets $X - f^{-1}(V)$. In the subspace topology for the space $f^{-1}[Cl(V)]$, there is an open basis element G containing x' such that $G \subseteq f^{-1}(V)$. Now $G = H \cap f^{-1}[Cl(V)]$, where H is open in X. Since X is locally connected, there is a connected open subset U of H containing x'. Since U is open in X and $x' \in U$, $U \cap [X - f^{-1}(V)] \neq \emptyset$. Thus, there is a point $x'' \in U$ such that $f(x'') \notin V$. Since $x'' \in H - G$, $f(x'') \notin Cl(V)$. So $f|_U$ maps the connected set U into the disconnected space $Y - Fr(V)$, but this is impossible since $f(U) \cap [Y - Cl(V)] \neq \emptyset$, $f(U) \cap V \neq \emptyset$, and $f(U) \cap V$ is clopen in the subspace $f(U)$. □

The following corollary generalizes a well-known result.

Corollary 2 *Let $f : R \to R$ be a real-valued function with derivative f'. Then f' is continuous if and only if f' is locally relatively continuous.*

5 Applications to Separate Continuity

In Corollary 3 we present a result involving separate continuity and local relative continuity. First, we recall the following.

Definition 2 *Consider a function $f : X \times Y \to Z$, where X, Y, and Z are topological spaces. For a given point (x', y') in the domain of f, the function $f_{x'}$ defined by $f_{x'}(y) = f(x', y)$, is called an x-section of f; and the function $f^{y'}$, given by $f^{y'}(x) = f(x, y')$, is called a y-section of f. If all x-sections and all y-sections of f are continuous, we say that f is separately continuous.*

Theorem 3 *Let X be a locally connected space, and let Y and Z be topological spaces. Suppose a function $f : X \times Y \to Z$ has continuous x-sections and connected y-sections. Then if f is locally relatively continuous, f is continuous.*

PROOF. There is an open basis $\mathbf{B}$ for the topology on Z such that for any $B \in \mathbf{B}$, $f^{-1}(B)$ is open in the subspace $f^{-1}[Cl(B)]$. It is sufficient to show that for any $B \in \mathbf{B}$, $f^{-1}(B)$ is open in $X \times Y$. Assume there exists a B in $\mathbf{B}$ such that $f^{-1}(B)$ is not open in $X \times Y$. Then there is a point $(x', y') \in f^{-1}(B)$ such that any open set in $X \times Y$ containing (x', y') meets $(X \times Y) - f^{-1}(B)$. In the subspace topology for $f^{-1}[Cl(B)]$ there exists an open basis element G containing (x', y') such that $G \subseteq f^{-1}(B)$. Now $G = G_0 \cap f^{-1}[Cl(B)]$, where G_0 is open in $X \times Y$. Thus, there is an open basis element U for the topology on X, and there is an open basis element V for the topology on Y such that $(x', y') \in (U \times V) \subseteq G_0$. Since f is continuous in y for every fixed x, there is an open set V_0 in Y such that $y' \in V_0 \subseteq V$ and for all $y \in V_0$, $f(x', y) \in B$. Since X is locally connected, there is a connected open set U_0 in X such that $x' \in U_0 \subseteq U$. Since $U_0 \times V_0$ is open in $X \times Y$ and contains (x', y'), then $(U_0 \times V_0) - f^{-1}(B) \neq \emptyset$. That is, there is a point $(x'', y'') \in U_0 \times V_0$ such that $f((x'', y'')) \notin B$. Also, since $(x'', y'') \notin G$, but is contained in G_0, $f((x'', y'')) \notin Cl(B)$. But this is impossible because $f^{y''}|_{U_0} : U_0 \to Z - Fr(B)$ is a connected function, and $f^{y''}|_{U_0}(U_0) \cap B$ is a nonempty clopen proper subset of $f^{y''}(U_0)$. □

Corollary 3 *Let X be a locally connected space. Then the function $f : X \times Y \to Z$ is continuous if and only if f is both separately continuous and locally relatively continuous.*

Remark 3 We are now in a position to show that the condition of local compactness on the space Y is essential in Corollary 1 above. Assume that Corollary 1 is true where the range space is Hausdorff but not locally compact. Then by Corollary 3, it follows that each separately continuous closed graph function $f : X \times Y \to Z$, where X is locally connected and Z is Hausdorff, is continuous; but this is shown to be impossible by Example 2 of [5].

In order to prove Theorem 4 below, we present the following.

Lemma 1 *Let $f : X \to R$ be a real-valued function. If f has closed fibers, then f is locally relatively continuous.*

PROOF. Consider the usual basis for R. Then f is locally $w*$ continuous and therefore locally relatively continuous. □

Remark 4 In the Lemma 1 above, if the space X is Hausdorff, then the statement holds if f has compact fibers.

Theorem 4 *Let $f : X \times Y \to R$ be a real-valued function, where X is a locally connected space. Suppose that f has continuous x-sections and connected y-sections. If f has closed fibers, then f is continuous.*

PROOF. See Lemma 1 and Theorem 3 above. □

Corollary 4 *A separately continuous real-valued function $f : R \times R \to R$ with compact (or closed) fibers is continuous.*

On the other hand, no separately continuous surjection from $R \times R$ into R can have compact fibers. By Corollary 4, such a function would be continuous. Now let $D_r = \{(x, y) \in R \times R \mid x^2 + y^2 \leq r^2\}$ and let $E_r = \{(x, y) \in R \times R \mid r^2 < x^2 + y^2\}$ for each $r > 0$. If $f : R \times R \to R$ is a continuous surjection with compact fibers, $f^{-1}(0)$ is compact and there exists $r_0 > 0$ such that $f^{-1}(0) \subseteq D_{r_0}$. Let $r = r_0 + 1$. Then since E_r is connected, $f(E_r)$ is an interval not containing 0. Further, being compact and connected, $f(D_r)$ is a closed and bounded interval, say $[a, b]$, containing 0. Finally, since f is surjective, $R = f(R \times R) = [a, b] \cup f(E_r)$, so that $a - 1 \in f(E_r)$ and $b + 1 \in f(E_r)$. But then $0 \in f(E_r)$ since $a - 1 < 0 < b + 1$, and $f(E_r)$ is an interval. This contradicts the fact that $0 \notin f(E_r)$.

6 Local Relative Continuity and Weak Continuity

Definition 3 *A function $f : X \to Y$ is weakly continuous at $x \in X$ if for any open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq Cl(V)$. If this condition is satisfied at each $x \in X$, then f is said to be weakly continuous.*

Because local relative continuity strictly generalizes local $w*$ continuity, the following decomposition of continuity improves Theorem 5 of [2].

Theorem 5 *A function $f : X \to Y$ is continuous if and only if it is both weakly continuous and locally relatively continuous.*

PROOF. The necessity is clear. Sufficiency: Let $\mathbf{B}$ be an open basis for the topology on Y such that $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$, for each $V \in \mathbf{B}$. By weak continuity, $f^{-1}(V) \subseteq Int\{f^{-1}[Cl(V)]\}$. Thus $f^{-1}(V)$ is open in the subspace $Int\{f^{-1}[Cl(V)]\}$, and hence is open in X, for $f^{-1}(V) = W \cap f^{-1}[Cl(V)]$ for some open set $W \subseteq X$ implies $f^{-1}(V) = W \cap Int\{f^{-1}[Cl(V)]\}$. Since preimages of basic open sets are open, f is continuous. □

Every function into a hyperconnected space is weakly continuous since open sets are dense in such a space. This yields the following.

Corollary 5 *If Y is hyperconnected, $f : X \to Y$ is continuous if and only if f is locally relatively continuous.*

A space Y is strongly locally countably compact if each point $y \in Y$ has a neighborhood whose closure is countably compact. Since such spaces need not be regular (even in the presence of Hausdorff separation), the following corollary strengthens Theorem 3.2 of [6]. Recall that a space X is Fréchet if whenever $x \in X$ is a limit point of a subset $A \subseteq X$, then there is a sequence of points in A converging to x. Also, a function $f : X \to Y$ is almost continuous (in the sense of Husain, [7]) if for each (basic) open set $V \subseteq Y$, $f^{-1}(V) \subseteq Int\{Cl[f^{-1}(V)]\}$.

Corollary 6 *Let $f : X \to Y$ be almost continuous where Y is a strongly locally countably compact space, and X is a Fréchet space. If the graph of f, $G(f)$, is closed then f is continuous.*

PROOF. Let $\mathbf{B}$ be an open basis for the topology on Y whose members have countably compact closures. By Theorem 2 of [2], for f to be weakly continuous, it is sufficient that $f^{-1}(V) \subseteq Int\{f^{-1}[Cl(V)]\}$ for each $V \in \mathbf{B}$. If $V \in \mathbf{B}$ and $x \in Cl\{f^{-1}[Cl(V)]\}$, there is a sequence of points (x_n) in $f^{-1}[Cl(V)]$ converging to x. Since $Cl(V)$ is countably compact, the sequence $(f(x_n))$ clusters at some point $y \in Cl(V)$. Evidently, (x, y) is a limit point of $G(f)$ which is closed so that $y = f(x)$. Thus, $x \in f^{-1}[Cl(V)]$, showing that $f^{-1}[Cl(V)]$ is closed for each $V \in \mathbf{B}$. This implies that $Int\{Cl[f^{-1}(V)]\} \subseteq Int\{f^{-1}[Cl(V)]\}$ for each $V \in \mathbf{B}$, so that f is weakly continuous. Further, f is locally relatively continuous and in fact, locally $w*$ continuous since $f^{-1}[Fr(V)]$ is closed for each $V \in \mathbf{B}$, since each $Fr(V)$ is countably compact, being closed in the countably compact space $Cl(V)$. By Theorem 5, f is continuous. □

Definition 4 *[9]A function $f : X \to Y$ is weakly α-continuous if $f^\alpha : X^\alpha \to Y$ is weakly continuous where the underlying set of the space X^α is X, a subset $A \subseteq X$ is open in X^α, or equivalently α-open in X, if and only if $A \subseteq Int\{Cl[Int(A)]\}$, and $f^\alpha(x) = f(x)$ for each $x \in X$.*

The condition of weak α-continuity is strictly weaker than that of weak continuity. ([9] and [8])

Theorem 6 *A function $f : X \to Y$ is continuous if and only if it is both weakly α-continuous and locally relatively continuous.*

PROOF. The necessity is clear. For the sufficiency, let **B** be an open base for the topology on Y such that $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$ for each $V \in \mathbf{B}$. Let τ be the topology on X and let τ^α be the α-topology for X. Then $\tau \subseteq \tau^\alpha$ and for each $V \in \mathbf{B}$, $f^{-1}(V)$ is open in $f^{-1}[Cl(V)]$, with $f^{-1}[Cl(V)]$ interpreted as a subspace of $(X, \tau^\alpha) = X^\alpha$. That is, $f^{-1}(V) = W \cap f^{-1}[Cl(V)]$ for some open set $W \in \tau \subseteq \tau^\alpha$. Now $f^\alpha : X^\alpha \to Y$, defined by $f^\alpha(x) = f(x)$ for each $x \in X$, is locally relatively continuous and also weakly continuous (since f is weakly α-continuous). By Theorem 5, f^α is continuous, and this implies [9] that f is weakly continuous. Again by Theorem 5, f is continuous. □

Remark 5 Again, because local relative continuity is strictly weaker than local $w*$ continuity, Theorem 6 improves Theorem 1 of [8]. A remark given in [8] shows that even if $f : X \to Y$ is weakly continuous, f may fail to be continuous even if $f^\alpha : X^\alpha \to Y$ is $w*$ continuous. Thus, local relative continuity for f cannot be replaced by local relative continuity for f^α in Theorem 6.

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