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SOME TOPOLOGICAL PROPERTIES OF HAMEL BASES

Abstract

We prove some results concerning covering, and translation properties of Hamel bases. For example, we show that the complement of a union of fewer than continuum many translates of a Hamel basis, more generally Erdős set, is everywhere of the second category.

1 Introduction

Several properties of Hamel bases have been extensively studied in [2]. All undefined notions used in this paper can be found in [2]. The real line R is a vector space over the rationals. Any basis for this vector space is called a Hamel basis. A subset X of the reals is said to be everywhere of the second category if the intersection of X with any nonempty open interval is not a countable union of nowhere dense subsets. Several properties of Hamel bases are studied in this paper. For example, we show that no Hamel basis is closed, more generally no Hamel basis is a countable union of closed sets and the complement of a union of fewer than continuum many translates of Erdős set $Z(H)$ is everywhere of the second category, where the Erdős set $Z(H)$ is defined to be the set of all finite linear combinations of elements from H with integer coefficients. This generalizes Theorem 7 in [3] " if D is a union of fewer than continuum many translates of a Hamel basis, then the complement of D is everywhere of the second category."

Theorem 1 *If K is a subset of R, K spans R over the rationals and no proper subset of K spans R, then K is not a countable union of closed sets.*

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PROOF. Suppose K is a countable union of closed sets. Then it can be easily seen that K is a countable union of compact sets K_i, where each K_i is a proper subset of K and $K_i \subseteq K_j$ for $i < j$. Let $S_i = r_1K_i + r_2K_i + ... + r_nK_i$, where $r_1, r_2, ..., r_n$ are fixed rational numbers. Since K_i is compact, it can be easily seen that S_i is closed. S_i is of measure zero and of the first category, otherwise $S_i - S_i$ would contain an interval [4, p.93] and hence S_i would span R contradicting K_i does not span R. Since R is a countable union of sets of the form S_i, R is a countable union of measure zero, first category sets, which is a contradiction. □

Corollary 1 *No Hamel basis is closed. (See [2, p.258].)*

If H is a Hamel basis, then there exists an element b such that $H+b$ contains no Hamel basis. This appears as an exercise in a book by M. Kuczma (An introduction to the theory of functional equations and inequalities, 1985). The following theorem gives a necessary and sufficient condition for $H + r$ to be a Hamel basis whenever H is a Hamel basis.

Theorem 2 *Let H be a Hamel basis and $r = \sum_{i=1}^{n} q_i h_i$, where q_i are rational numbers and h_i are elements of H. Then $H + r$ is a Hamel basis if and only if $\sum_{i=1}^{n} q_i \neq -1$.*

PROOF. Suppose $H + r$ is a Hamel basis. Then $r = \sum_{i=1}^{m} b_i(h_i + r)$, where b_i are rational numbers and h_i are elements of H. Hence $\sum_{i=1}^{m} b_i h_i = (1 - \sum_{i=1}^{m} b_i)r = (1 - \sum_{i=1}^{m} b_i) \sum_{i=1}^{n} q_i h_i$. Without loss of generality we may assume that $n = m$. Since H is a Hamel basis, $b_j = (1 - \sum_{i=1}^{m} b_i)q_j$ for $1 \leq j \leq n$. Consequently $\sum_{j=1}^{n} b_j = \sum_{j=1}^{n}(1 - \sum_{i=1}^{m} b_i)q_j = (1 - \sum_{i=1}^{m} b_i)(\sum_{j=1}^{n} q_j)$. Hence $\sum_{j=1}^{n} q_j \neq -1$, otherwise $\sum_{j=1}^{m} b_j = (\sum_{i=1}^{m} b_i) - 1$, which is impossible.

Conversely, assume that $\sum_{i=1}^{n} q_i \neq -1$. Suppose $\sum_{i=1}^{m} b_i(h_i+r) = 0$, where b_i are rational numbers and h_i are elements of H. Then $0 = \sum_{i=1}^{m} b_i h_i + (\sum_{i=1}^{m} b_i)(\sum_{i=1}^{n} q_i h_i)$ (*). Without loss of generality we may assume that $n = m$. By (*), $b_j + (\sum_{i=1}^{n} b_i)q_j = 0$ for $1 \leq j \leq n$. Since $0 = \sum_{j=1}^{n}[b_j + (\sum_{i=1}^{n} b_i)q_j] = (\sum_{i=1}^{n} b_i)(1+ \sum_{i=1}^{n} q_i)$ and $\sum_{i=1}^{n} q_i \neq -1, \sum_{i=1}^{n} b_i = 0$. Now, because of $b_j + (\sum_{i=1}^{n} b_i)q_j = 0$, $b_j = 0$. Thus $H + r$ is a Hamel basis. □

Remark 1 *It follows from the above proof that if H is a Hamel basis, then $H + r$ contains a Hamel basis if and only if $H + r$ is a Hamel basis.*

Sierpinski proved that the complement of a Hamel basis is everywhere of the second category and Morgan [3, Th. 7] generalized that if H is a Hamel basis and D is a union of fewer than continuum many translates of H, then the complement of D is everywhere of the second category. The following theorem is little more than the above result.

Theorem 3 *If D is a union of fewer than continuum many translates of an Erdős set $Z(H)$, where H is a Hamel basis, then the complement of D is everywhere of second category.*

PROOF. Since $Z(H)$ is an additive subgroup of the reals R and the index of $Z(H)$ in R is the cardinality of the continuum, D is contained in a proper subgroup of R, namely the sum of $Z(H)$ and a subgroup of cardinality less than the cardinality of the continuum. The proof is complete if we show that the complement of a proper subgroup of R is everywhere of the second category. To do this, let G be a proper subgroup of R and suppose that the complement of G is not everywhere of the second category. Then there exists a nonempty open interval I such that $(R\backslash G) \cap I$ is of the first category and hence $G = G - G \supseteq (G \cap I) - (G \cap I)$ contains a nonempty open interval, see [4]. Hence $G = R$, which is a contradiction. Thus the proof is complete. □

It is well known that R is not a direct sum of a proper subgroup G and the group Z of integers. Is R the sum of G and Z? To answer the question, suppose $R = G + Z$. Then for every nonzero integer n, there exist $g_n \in G$ and $z_n \in Z$ such that $\frac{1}{n} = g_n + z_n$. Hence $1 - nz_n \in G \cap Z$. Let $a = 1 - 2z_2$. Then a and $1 - az_a$ are in $G \cap Z$. Hence $1 = (1 - az_a) + az_a \in G \cap Z$. This implies that $G \cap Z = Z$ and $R = G + Z \subseteq G + G$, which contradicts that G is a proper subgroup of R. Thus $R \neq G + Z$. However it follows from the following theorem that R is a sum of a proper subgroup and an Erdős subgroup $Z(H)$, when H is a Hamel basis.

Theorem 4 *There exist a proper subgroup G of R and a Hamel basis H of R such that $R = G + H_1$ for a countably infinite subset H_1 of H.*

PROOF. Let B be a Hamel basis and let b be a fixed element of B. Then $E(B\backslash\{b\})$ (the set of all finite linear combinations of elements from $B\backslash\{b\}$ with rational coefficients) is a proper subgroup of R. Well-order $B\backslash\{b\}$ as $(b_\xi)_{\xi<\Omega}$ and well-order the set of all rationals as $(q_\xi)_{\xi<\omega}$. Let $H_1 = \{q_\xi b + b_\xi : 1 \leq \xi < \omega\}$. Then it can be easily seen that H_1 is linearly independent and hence there exists a Hamel basis H containing H_1. Now $R = E(B\backslash\{b\}) + H_1$ because any real number r can be written as a finite linear combination of elements from B with rational coefficients, that is, there exists $g \in E(B\backslash\{b\})$ such that $r = g + q_\xi b = (g - b_\xi) + (q_\xi b + b_\xi) \in E(B\backslash\{b\}) + H_1$. □

It is interesting to compare the following theorem with Corollary 2.

Theorem 5 *The complement of a finite union of Hamel bases is everywhere of the second category.*

PROOF. Suppose $R\backslash \bigcup_{i=1}^{n} H_i$, where H_i's are Hamel bases, is not everywhere of the second category. Then there exists a nonempty open interval I such that $(R\setminus \cup_{i=1}^{n} H_i) \cap I$ is of the first category (*). Hence $H_i \cap I$ is of the second category for some i, say $i = 1$. Let r be a real number such that $0 \in I + r$. Then for every $h \in H_1$, there exists a positive integer m such that $\frac{h}{m} \in I + r$. Since H_1 is of the second category and countable union of the first category sets is of the first category, there exists a positive integer m such that $\{h \in H_1 : \frac{h}{m} \in I + r\}$ is of the second category. Let m be a positive integer such that $\{h \in H_1 : \frac{h}{m} \in I + r\}$ is of the second category and let $H' = \{h \in H_1 : \frac{h}{m} \in I + r\}$ (**). Then, since $I + r$ is an open interval containing zero, for every integer $q \geq m$, $\frac{h}{q} \in I + r$ for all h in H' and by (**), $\frac{H'}{q} = \{\frac{h}{q} : h \in H'\}$ is of the second category.

We shall prove that, for every $h \in H'$ there are only finitely many prime numbers q for which $\frac{h}{q} \in (\cup_{i=1}^{n} H_i \cap I) + r$. Suppose for some $h \in H'$ and for some i, $1 \leq i \leq n$, $\frac{h}{q} \in (H_i \cap I) + r$ for infinitely many prime numbers q. Then there are prime numbers s, t, u, v with $s < t < u < v$ and some elements a, b, c, d of H_i such that $\frac{h}{s} = a+r$, $\frac{h}{t} = b+r$, $\frac{h}{u} = c+r$ and $\frac{h}{v} = d+r$. This implies that $st\frac{a-b}{t-s} = uv\frac{c-d}{v-u}$ and since $\{a, b, c, d\}$ is linearly independent, $\frac{st}{t-s} = \frac{uv}{v-u}$. Hence v divides $st(v-u)$, which is impossible. Thus, for every $h \in H'$, there are only finitely many prime numbers q for which $\frac{h}{q} \in (\cup_{i=1}^{n} H_i \cap I) + r$. Now since $I + r = [\cup_{i=1}^{n}(H_i \cap I) + r] \cup [((R\setminus \cup_{i=1}^{n} H_i) \cap I) + r]$, for every $h \in H'$, there exists a prime number p such that $\frac{h}{q} \in [(R\setminus \cup_{i=1}^{n} H_i) \cap I] + r$ for all prime numbers $q \geq p$. Again, since H' is of the second category and countable union of the first category sets is of the first category, there exists a prime number p such that $\{h \in H' : \frac{h}{p} \in [(R\setminus \cup_{i=1}^{n} H_i) \cap I] + r\}$ is of the second category. This contradicts (*) and thus the proof is complete. □

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