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Luisa Di Piazza*, Istituto Matematico, Università di Palermo, via Archirafi, 34, 90123 Palermo, Italy, email: dipiazza@ipamat.math.unipa.it

A NOTE ON ADDITIVE FUNCTIONS OF INTERVALS

Abstract

If F is a continuous function of intervals in $\mathbb{R}^m$, then its distribution function is continuous. The converse is true if $m = 1$ but false if $m \geq 2$. In the present note we prove these facts and we explain why the one-dimensional case is an exception.

An *interval* is always a nonempty compact interval in $\mathbb{R}^m$, i.e., the product

$$[a_1, b_1; \ldots; a_m, b_m] = \prod_{i=1}^{m} [a_i, b_i]$$

where $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$ for $i = 1, \ldots, m$. A *figure* is the union of a nonempty finite family of intervals. The closure, interior, boundary, and m-dimensional Hausdorff measure $\mathcal{H}^m$ of a figure $A \subset \mathbb{R}^m$ is denoted by A^-, A°, ∂A and $|A|$, respectively; the *perimeter* of A is the $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}$ of its *essential boundary* $\partial^*(A) = \partial[(A^-)^\circ]$, and it is denoted by $\|A\|$. A figure A with $|A| = 0$ (equivalently, $A^\circ = \emptyset$ or $\partial^*(A) = \emptyset$) is called *degenerate*. We say figures A and B *overlap* if $A \cap B$ is nondegenerate.

Throughout this note, we select a fixed interval $A = [a_1, b_1; \ldots; a_m, b_m]$. A function F defined on the family of all subfigures of A is called an *additive function* in A whenever

$$F(B \cup C) = F(B) + F(C)$$

for each pair B, C of nonoverlapping subfigures of A. Clearly, each additive function in A vanishes on every degenerate subfigure of A. If F is an additive function in A and $x = (x_1, \ldots, x_m)$ is a point in A, we let

$$f(x) = F([a_1, x_1; \ldots; a_m, x_m])$$

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and call the function $f : x \mapsto f(x)$, defined on A, the *distribution function* of F. A standard calculation shows that for each interval $[c_1, d_1; \ldots; c_m, d_m] \subset A$, we obtain

$$F([c_1, d_1; \ldots; c_m, d_m]) = \sum (-1)^{\sigma(x)} f(x)$$

where the summation is taken over all points $x = (x_1, \ldots, x_m)$ such that for $i = 1, \ldots, m$, either $x_i = c_i$ or $x_i = d_i$, and $\sigma(x)$ is the cardinality of the set $\{i : x_i = c_i\}$. Since F is uniquely determined by its values on intervals, it is also uniquely determined by its distribution function.

In the theory of conditionally convergent integrals, a prominent role is played by additive functions that are continuous in the following sense (cf. [1, Section 11.2]).

Definition 1 An additive function F in A is *continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

It is easy to see that the distribution function of a continuous additive function in A is continuous. The converse is true if $m = 1$ but false if $m \geq 2$. We prove these facts, and explain why the one-dimensional case is an exception.

Proposition 2 *Assume $m = 1$, and let f be the distribution function of an additive function F in A. If f is continuous, then so is F.*

PROOF. Choose an $\varepsilon > 0$ and use the uniform continuity of f to find an $\eta > 0$ so that $|f(x) - f(y)| < \varepsilon^2$ for each $x, y \in A$ with $|x - y| < \eta$. If B is a subfigure of A, then $B = \bigcup_{k=1}^{n} [c_k, d_k]$ where $c_1 \leq d_1 < \cdots < c_n \leq d_n$ are points of A, and $\|B\|$ equals twice the number of nondegenerate intervals $[c_k, d_k]$. Thus

$$F(B) = \sum_{k=1}^{n} [f(d_k) - f(c_k)] < \varepsilon^2 \|B\| < \varepsilon$$

whenever $\|B\| < 1/\varepsilon$ and $|B| < \eta$. □

Example 3 We assume $m = 2$; the construction for $m > 2$ is similar. Let $A = [0,1]^2$, and for $k = 1, 2, \ldots$ and $t \in [0,1]$, set

$$f(t,0) = f(0,t) = f(2^{-k}, 2^{-k}) = 0 \quad \text{and} \quad f(2^{-k+1}, 2^{-k}) = 1/k \, .$$

Since f is a continuous function on a closed set

$$C = \{(t,0), (0,t), (2^{-k}, 2^{-k}), (2^{-k+1}, 2^{-k}) : t \in [0,1], k = 1, 2, \ldots\}$$

contained in A, it has a continuous extension to the whole of A, still denoted by f. Define an additive function in A by setting

$$F([a,b]\times[c,d]) = f(a,c) + f(b,d) - f(a,d) - f(b,c)$$

for each interval $[a,b]\times[c,d] \subset A$, and observe that f is the distribution function of F. To see that F is not continuous, let $A_k = [2^{-k}, 2^{-k+1}]\times[0,2^{-k}]$ for $k = 1,2,\dots$. As

$$\sum_{k=1}^{\infty} F(A_k) = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty\,,$$

for each integer $n \geq 1$ there is a integer $p_n \geq n$ such that $\sum_{k=1}^{p_n} F(A_k) > 1$. If $B_n = \bigcup_{k=n}^{p_n} A_k$, then

$$\begin{aligned} \|B_n\| &< 4\sum_{k=n}^{\infty} 2^{-k} = 8\cdot 2^{-n}\,, \\ |B_n| &< \sum_{k=n}^{\infty} 2^{-2k} = \frac{4}{3}\cdot 4^{-n} \end{aligned}$$

for $n = 1,2,\dots$. It follows that F is, indeed, discontinuous.

In dimension one, the connection between an additive function F in A and its distribution function f can be cast differently. For a figure $B \subset A$, denote by ν_B its *exterior unit normal*, i.e., the function associating to each $x \in \partial^* B$ the number $+1$ or -1 according to whether x is the right or left end-point of a nondegenerate connected component of B. Now viewing f as a vector field in A, we see that $F(B) = \int_{\partial^* B} f\cdot\nu_B\, d\mathcal{H}^0$. With this interpretation of f, the next proposition (cf. [1, Proposition 11.2.8]) illuminates Proposition 2.

Proposition 4 *Let v be a continuous vector field in A, and for each figure $B \subset A$ let $F(B) = \int_{\partial^* B} v\cdot\nu_B\, d\mathcal{H}^{m-1}$. Then F is a continuous additive function in A.*

PROOF. As the additivity of F is clear, choose an $\varepsilon > 0$ and find a vector field w whose coordinates are polynomials and such that $|v(x) - w(x)| < \varepsilon^2/2$ for all $x \in A$. Let α be a positive bound of $|\operatorname{div} w|$ on A, and set $\eta = \varepsilon/(2\alpha)$. If $B \subset A$ is a figure with $\|B\| < 1/\varepsilon$ and $|B| < \eta$, the divergence theorem and Schwartz inequality give

$$\begin{aligned} |F(B)| &\leq \left|\int_{\partial^*(B)} (v-w)\cdot\nu_B\, d\mathcal{H}^{m-1}\right| + \left|\int_B \operatorname{div} w\, d\mathcal{H}^m\right| \\ &\leq \frac{\varepsilon^2}{2}\|B\| + \alpha|B| < \varepsilon\,, \end{aligned}$$

end the proof is completed. □

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References

[1] W. F. Pfeffer, *The Riemann Approach to Integration*, Cambridge Univ. Press, Cambridge, 1993.