Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 805–814

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DARBOUX HOMOTOPIES AND DARBOUX RETRACTS - RESULTS AND QUESTIONS

The theory of Darboux functions is still being studied intensely by many mathematicians. The notion of a Darboux function itself has gone through lots of generalizations. The best-known generalization is included in the following definition:

> We say that $f : X \to Y$, where X and Y are arbitrary topological spaces, is a Darboux transformation if $f(C)$ is a connected set for each connected set $C \subset X$.

We shall apply such a definition at present. It seems that the theory of Darboux functions has not, however, been extended widely enough so far, although the former interest of mathematicians in these transformations would justify that to the full. What hinders one from extending it is the lack of good tools for

> the examination of algebraic and topological structures of the space formed by Darboux functions.

Analyzing the methods for the investigation of the space of continuous functions, one can come to the conclusion that the foundations of the considerations carried out there are: rings of functions, retracts and homotopies (as well as related theories and notions such as, for instance, ideals of rings of functions, contractibility, domination, and the like). Consequently, it seems interesting to build analogous structures and theories for Darboux transformations.

In this article we would like to present our results as well as the basic open problems and questions which are connected with the building of this theory.

We apply the classical symbols and notions. In particular, the symbol $[0, 1]$ denotes the unit interval with the natural topology. By the letters $\Re$, N, Q

Mathematical Reviews subject classification: Primary: 26A15 Secondary: 54C40 54C30

Received by the editors September 15, 1994

we denote the set of all real numbers, positive integers and rational numbers, respectively. If (X, T) is some topological space, then by $I(X, T)$ we denote the set of all isolated points. Let X, Y be topological spaces; then $D(X, Y)$ denotes the set of all Darboux functions mappings X into Y. If F is some family of functions mappings X into Y, then $C_F = \bigcap_{f \in F} C_f$ where C_f stands for the set of all points of continuity of f. For $A \subset X \times Y$, the symbol $proj_X(A)$ denotes the projection of A onto X.

Now, we shall introduce three distinct generalizations of the notion of a homotopy. Thanks to this, we shall make it possible to consider a wide range of algebraic and topological problems as well as of problems from the borderland of real function theory and algebraic topology.

Definition. Let $f, g : (X, T) \to (Y, D)$ be Darboux functions, where (X, T), (Y, D) are topological spaces.
(a) The transformations f and g are called **weakly Darboux homotopic** [1] if there exists a Darboux transformation $\xi : (X, T) \times [0, 1] \to (Y, D)$ such that $\xi((x, 0)) = f(x)$ and $\xi((x, 1)) = g(x)$ for $x \in X$. The transformation ξ is called a **weak Darboux homotopy** between f and g. (The fact that f and g are weakly Darboux homotopic and ξ is a weak Darboux homotopy between f and g is written down as $f \overset{wDbx(\xi)}{\simeq} g$, or shortly $f \overset{wDbx}{\simeq} g$).
(b) The transformations f and g are called **Darboux homotopic** if $f \overset{wDbx(\tau)}{\simeq} g$ and each thread $\tau|_{\{x\} \times [0,1]}$ is a continuous function (for $x \in X$). (The fact that f and g are Darboux homotopic and τ is a Darboux homotopy between f and g is written down as $f \overset{Dbx(\tau)}{\simeq} g$, or shortly $f \overset{Dbx}{\simeq} g$).
(c) The transformations f and g are called **strongly Darboux homotopic** if $f \overset{Dbx(\tau)}{\simeq} g$ and there exists topology T^* in the set X finer than the topology T for which $I(X, T) = I(X, T^*)$, such that $f, g : (X, T^*) \to (Y, D)$ are continuous and homotopic. (The fact that f and g are strong Darboux homotopic is written down as $f \overset{sDbx}{\simeq} g$).

Note that if X, Y are arbitrary topological spaces and $f, g : X \to Y$, then:
(i) if f, g are continuous and homotopic, then $f \overset{sDbx}{\simeq} g$;
(ii) if $f \overset{sDbx}{\simeq} g$, then $f \overset{Dbx}{\simeq} g$;
(iii) if $f \overset{Dbx}{\simeq} g$, then $f \overset{wDbx}{\simeq} g$.
Of course, the inverse implications are not true. In particular, if $X = \{x_0\}$ is a singleton and $Y = \{(0, 0)\} \cup \{(x, y) : x > 0 \wedge y = sin\ \frac{1}{x}\}$ with the natural topology, $f(x_0) = (0, 0)$ and $g(x_0) = (1, sin\ 1)$, then f and g are continuous

[1] In the terminology of paper [9] - *c-homotopic*

and weakly Darboux homotopic but f and g are not Darboux homotopic and, consequently, f and g are neither strongly Darboux homotopic nor homotopic.

Theorem 1 ([9]) *Let $f, g : X \to Y$ be continuous functions, where X is a contractible and separable space and Y is a compact space. If*
$f(X) \cap g(X) \neq \emptyset$
or
$f(X) \cap g(X) = \emptyset$ *and* $f \overset{wDbx(\zeta)}{\simeq} g$ *and*
ζ is a weak Darboux homotopy such that there exists $x_0 \in X$ for which $[\zeta(\{x_0\} \times [0,1])]^d \setminus [\zeta(\{x_0\} \times [0,1])] = \{y_1, y_2\}$ where $y_1 \in f(X), y_2 \in g(X)$ and, for each $\alpha \in \zeta(\{x_0\} \times [0,1])$, α cuts Y between $f(X)$ and $g(X)$, then f and g are homotopic.

Problem 1 *The assumption in the above theorem are complicated and connected with the notion of a Darboux homotopy. It is interesting to give, for instance,*
general topological conditions on the spaces under consideration in order that the implication inverse to the implications (i), (ii) or (i) be true. It is especially interesting to state the conditions under which strong Darboux homotopies are equivalent to Darboux ones.

Algebraic operations performed on Darboux functions are the objects of interest of many authors ([1], [3], [4], [5], [7], [10], [12]). Lots of questions connected with this topic are still unsolved. Among them, particularly interesting are those which concern the question of when two Darboux functions belong to a common complete ring of Darboux functions [2]. It turns out that, when one studies these problems, very helpful is the theory of Darboux homotopies.

Theorem 2 *Let $\mathcal{P}$ be a complete ring of Darboux functions mapping $[0,1]$ into $\Re$, such that $C_{\mathcal{P}}$ is an open and dense set, $f(x) = 0$ for $f \in \mathcal{P}$ and $x \in D_{\mathcal{P}}$. Then any two functions from $\mathcal{P}$ are strongly Darboux homotopic.*

Proof. Let $f, g \in \mathcal{P}$ and let $C_{\mathcal{P}} = \{(a_n, b_n) : n \in N\}$ where (a_n, b_n) denotes a component of $C_{\mathcal{P}}$ (of course, if, for example, $0 \in C_{\mathcal{P}}$, then, for some n_0, (a_{n_0}, b_{n_0}) denotes the interval $[0, b_{n_0})$; to simplify the notation, we use the symbol (a_n, b_n) for all components of $C_{\mathcal{P}}$). Fix a positive integer n, let $f_n = f_{|[a_n,b_n]}$, $g_n = g_{|[a_n,b_n]}$ and let
c_n be the middle point of $[a_n, b_n]$. Let $\xi = | f | + | g |$. Then $\xi_n = \xi_{|[a_n,b_n]} = | f_n | + | g_n |$. Of course, $\xi \in \mathcal{P}$, and so, ξ_n is a Darboux function continuous on the interval (a_n, b_n).

[2]A ring $\mathcal{P}$ of functions is called a complete ring if $| f | \in \mathcal{P}$ for each $f \in \mathcal{P}$

Let $s_1^{(n)} \in (a_n, c_n)$, $z_1^{(n)} \in (c_n, b_n)$ be points such that $\xi(s_1^{(n)}), \xi(z_1^{(n)}) \in [0,1)$ and let $\delta_1^{(n)}, \sigma_1^{(n)} > 0$ be real numbers such that $a_n < s_1^{(n)} - \delta_1^{(n)} < s_1^{(n)} < s_1^{(n)} + \delta_1^{(n)} < c_n < z_1^{(n)} - \sigma_1^{(n)} < z_1^{(n)} < z_1^{(n)} + \sigma_1^{(n)} < b_n$, and $\xi((s_1^{(n)} - \delta_1^{(n)}, s_1^{(n)} + \delta_1^{(n)})) \subset [0,1)$ and $\xi((z_1^{(n)} - \sigma_1^{(n)}, z_1^{(n)} + \sigma_1^{(n)})) \subset [0,1)$. Now, let $s_2^{(n)} \in (a_n, min(s_1^{(n)} - \delta_1^{(n)}, a_n + \frac{1}{2}(s_1^{(n)} - a_n)))$, $z_2^{(n)} \in max(z_1^{(n)} + \sigma_1^{(n)}, b_n - \frac{1}{2}(b_n - z_1^{(n)})), b_n)$ be points such that $\xi(s_2^{(n)}), \xi(z_2^{(n)}) \in [0, \frac{1}{2})$ and let $\delta_2^{(n)}, \sigma_2^{(n)} > 0$ be real numbers such that $a_n < s_2^{(n)} - \delta_2^{(n)} < s_2^{(n)} < s_2^{(n)} + \delta_2^{(n)} < min\ (s_1^{(n)} - \delta_1^{(n)}, a_n + \frac{1}{2}(s_1^{(n)} - a_n)) < max(z_1^{(n)} + \sigma_1^{(n)}, b_n - \frac{1}{2}(b_n - \frac{1}{2}(b_n - z_1^{(n)})) < z_2^{(n)} - \sigma_2^{(n)} < z_2^{(n)} <$
$z_2^{(n)} + \sigma_2^{(n)} < b_n$, and $\xi((s_2^{(n)} - \delta_2^{(n)}, s_2^{(n)} + \delta_2^{(n)})) \subset [0, \frac{1}{2})$, $\xi((z_2^{(n)} - \sigma_2^{(n)}, z_2^{(n)} + \sigma_2^{(n)})) \subset [0, \frac{1}{2})$. Continuing this procedure we obtain two sequences: $\{s_k^{(n)}\}_{k=1}^{\infty}$ and $\{z_k^{(n)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} s_k^{(n)} = a_n, \lim_{k\to\infty} z_k^{(n)} = b_n$ and, moreover, we obtain sequences of pairwise disjoint intervals $\{(s_k^{(n)} - \delta_k^{(n)}, s_k^{(n)} + \delta_k^{(n)})\}_{k=1}^{\infty}$, $\{z_k^{(n)} - \sigma_k^{(n)}, z_k^{(n)} + \sigma_k^{(n)})\}_{k=1}^{\infty}$ such that $\xi((s_k^{(n)} - \delta_k^{(n)}, s_k^{(n)} + \delta_k^{(n)})) \subset [0, \frac{1}{k}) \supset \xi((z_k^{(n)} - \sigma_k^{(n)}, z_k^{(n)} + \sigma_k^{(n)}))$ (for $k = 1, 2, \ldots$). Note that

$$f((s_k^{(n)} - \delta_k^{(n)}, s_k^{(n)} + \delta_k^{(n)})) \subset (-\frac{1}{k}, \frac{1}{k}) \supset f((z_k^{(n)} - \sigma_k^{(n)}, z_k^{(n)} + \sigma_k^{(n)})),$$

$$g((s_k^{(n)} - \delta_k^{(n)}, s_k^{(n)} + \delta_k^{(n)})) \subset (-\frac{1}{k}, \frac{1}{k}) \supset g((z_k^{(n)} - \sigma_k^{(n)}, z_k^{(n)} + \sigma_k^{(n)})),$$

for $k = 1, 2, \ldots$.

To simplify the notation, by $\frac{1}{m}A_k^{(n)}$ ($\frac{1}{m}B_k^{(n)}$) we shall denote the interval $(s_k^{(n)} - \frac{1}{m}\delta_k^{(n)}, s_k^{(n)} + \frac{1}{m}\delta_k^{(n)})$ $((z_k^{(n)} - \frac{1}{m}\sigma_k^{(n)}, z_k^{(n)} + \frac{1}{m}\sigma_k^{(n)}))$ for $m, n, k \in N$ (of course, instead of $1A_k^{(n)}$ ($1B_k^{(n)}$) we shall write $A_k^{(n)}$ ($B_k^{(n)}$)).

Now, we shall define the family of neighbourhoods of x where $x \in [0,1]$.

If $x \in (a_n, b_n) \in C_{\mathcal{P}}$, then put $B_m(x) = (x - \frac{1}{m}, x + \frac{1}{m}) \cap (a_n, b_n)$ for $m = 1, 2, \ldots$.

Now, let $x \in D_{\mathcal{P}}$. Then let (for $m = 1, 2, \ldots$)
$B_m^-(x) = (x - \frac{1}{m}, x) \cap \left(\bigcup_n \left(\bigcup_{k=m}^{\infty} \frac{1}{m}B_k^{(n)}\right)\right)$ (in the case when $0 \in D_{\mathcal{P}}$, $B_m^- = \emptyset$);
$B_m^+(x) = (x, x + \frac{1}{m}) \cap \left(\bigcup_n \left(\bigcup_{k=m}^{\infty} \frac{1}{m}A_k^{(n)}\right)\right)$ (in the case when $1 \in D_{\mathcal{P}}$, $B_m^+(1) = \emptyset$).
Finally, we put $B_m(x) = B_m^-(x) \cup \{x\} \cup B_m^+(x)$ ($m = 1, 2, \ldots$).

Let $\mathcal{T}$ be a topology generated by the neighbourhood system

$$\{\{B_m(x) : m = 1, 2, \ldots\}\}_{x \in [0,1]}.$$

Of course, $\mathcal{T}$ is finer than the natural topology of $[0,1]$, $I([0,1],\mathcal{T}) = \emptyset$ and, moreover, $([0,1],\mathcal{T})$ is a Hausdorff and Fréchet space ([2]). It is not hard to verify that $f, g : ([0,1],\mathcal{T}) \to \Re$ are continuous functions.

Since $\mathcal{P}$ is a ring of functions, there exists a function $h \in \mathcal{P}$ such that $f + h = g$. Then $h : ([0,1],\mathcal{T}) \to \Re$ is a continuous function and $h(x) = 0$ for $x \in D_{\mathcal{P}}$.

Define the function $\tau : [0,1] \times [0,1] \to \Re$ in the following way:

$$\tau(x,t) = f(x) + t \cdot h(x).$$

It is evident that $\tau(x,0) = f(x)$ and $\tau(x,1) = g(x)$ (for $x \in [0,1]$).

Now, we shall show that $\tau : ([0,1],\mathcal{T}) \times [0,1] \to \Re$ is a homotopy. It is sufficient to show, that τ is a continuous transformation.

Let $(x_0, t_0) \in [0,1] \times [0,1]$ and let $\{(x_p, t_p)\}_{p=1}^{\infty}$ be a sequence[3] such that $\lim_{p\to\infty}(x_p, t_p) = (x_0, t_0)$. It is easy to see that $x_0 \in \mathcal{T} - \lim_{p\to\infty} x_p$ and $t_0 = \lim_{p\to\infty} t_p$ (where $\mathcal{T}-\lim_{p\to\infty} x_p$ denotes the limit of $\{x_p\}_{p=1}^{\infty}$ in the space $([0,1],\mathcal{T})$). Then $\lim_{p\to\infty} f(x_n) = f(x_0)$, $\lim_{p\to\infty} g(x_n) = g(x_0)$ and, at the same time, $\tau(x_p, t_p) \to \tau(x_0, t_0)$, which means that $\tau : ([0,1],\mathcal{T}) \times [0,1] \to \Re$ is a homotopy.

Now, we shall show that $\tau : [0,1] \times [0,1] \to \Re$ is a Darboux homotopy. First, we notice that each thread $\tau|_{\{x_0\}\times[0,1]}$ is a linear function, thus it is continuous.

To finish the proof, it is sufficient to show that $\tau : [0,1] \times [0,1] \to \Re$ is a Darboux function.

Let C be an arbitrary connected set in $[0,1] \times [0,1]$. Of course, if $C \subset \{x_0\} \times [0,1]$ (for some $x_0 \in [0,1]$) or $C \subset (a_n, b_n) \times [0,1]$ (for some n), then $\tau(C)$ is a connected set. Now, we suppose that the above cases do not take place and assume to the contrary that $\tau(C)$ is a disconnected set, which means that $\tau(C) = P \cup Q$ where P and Q are nonempty separated sets. Denote $P_0 = C \cap \tau^{-1}(P), Q_0 = C \cap \tau^{-1}(Q)$. Then P_0, Q_0 are nonempty disjoint sets such that $C = P_0 \cup Q_0$, and so, P_0, Q_0 are not separated sets. Let, for instance,

$$(1) \qquad (q_0, d_0) \in \overline{P_0} \cap Q_0.$$

In the case when $q_0 \in C_{\mathcal{P}}$, (q_0, d_0) is a continuity point of τ and, consequently $Q \ni \tau(q_0, d_0) \in \overline{\tau(P_0)}$, which contradicts the fact that P and Q are separated.

Consequently, we may infer that $q_0 \in D_{\mathcal{P}}$ and $\tau(q_0, d_0) = 0$.

[3]The fact that, in this case, it suffices to consider such ("countable") sequences follows from Theorem
1.6.14 and Proposition 1.6.15 [2] ($([0,1],\mathcal{T}) \times [0,1]$ is a Fréchet space because it is the Cartesian product of the first countable space).

According to the assumptions connected with the properties of the set C, we may deduce that $proj_X(C)$ is not included in $\{q_0\}$. Moreover, it is easy to see that P_0 is not a subset of $D_{\mathcal{P}} \times [0,1]$. So, let (a_{n_0}, b_{n_0}) be a component of $C_{\mathcal{P}}$ such that $((a_{n_0}, b_{n_0}) \times [0,1]) \cap P_0 \neq \emptyset$ and let $(w_0, l_0) \in P_0 \cap ((a_{n_0}, b_{n_0}) \times [0,1])$. Fix a positive integer k_0 such that $s_{k_0}^{(n_0)} < w_0 < z_{k_0}^{(n_0)}$ and $w_0 \notin A_{k_0}^{(n_0)} \cup B_{k_0}^{(n_0)}$. We shall show that

$$(2) \qquad P_0 \cap A_k^{(n_0)} \neq \emptyset \ \text{ or } \ P_0 \cap B_k^{(n_0)} \neq \emptyset \ \text{ for each } k \geq k_0.$$

Suppose to the contrary that:

$$(3) \qquad \text{there exists } k^* \geq k_0 \text{ such that } P_0 \cap A_{k^*}^{(n_0)} = \emptyset = P_0 \cap B_{k^*}^{(n_0)}.$$

Denote $M = P_0 \cap ([s_{k^*}^{(n_0)} + \delta_{k^*}^{(n_0)}, z_{k^*}^{(n_0)} - \sigma_{k^*}^{(n_0)}] \times [0,1]) \neq \emptyset$. Consider the sets M and $C \setminus M$. It is easy to see that $M \neq \emptyset \neq C \setminus M$ and $C = M \cup (C \setminus M)$. From (3), the continuity of $\tau|_{(a_{n_0}, b_{n_0}) \times [0,1]}$ and the fact that P, Q are separated sets we may infer that $\overline{M} \cap (C \setminus M) = \emptyset = M \cap \overline{C \setminus M}$, which contradicts the connectedness of C. The proof of (2) is finished.

From (2) we deduce that:

> there exists a sequence $\{x_m, t_m)\} \subset P_0$ such that $x_m \in A_{k_m}^{(n_0)}$ (and $k_m \to \infty$)

or

> there exists a sequence $\{y_m, d_m)\} \subset P_0$ such that $y_m \in B_{k_m}^{(n_0)}$ (and $k_m \to \infty$).

Suppose that the first of the above possibilities takes place (in the case of the second, the proof is similar). Then $\tau\ (x_m, t_m) \to \ 0 = \tau(q_0, d_0)$. Note that $\tau(x_m, t_m) \in P$ and, according to (1), $\tau(q_0, d_0) \in Q$, which is impossible because P and Q are separated sets.

The contradiction obtained proves that $\tau : [0,1] \times [0,1] \to \Re$ is a Darboux transformation, which ends the proof of the theorem.

Problem 2 *The analysis of concrete examples allows one to suppose that the converse theorem is also true. As yet, however, one has not managed to find a proof in the general case or an appropriate counterexample. A "style" of the conversion of this theorem is represented by the theorem below (concerning the case with very strong assumptions on the transformations considered).*

Theorem 3 *Let $f, g : [0,1] \to \Re$ be Darboux functions such that $C_f = C_g \supset (0,1]$ and $f(0) = g(0) = 0$. Then the functions f and g belong to some complete ring $\mathcal{P}$ of Darboux functions*[4] *if and only if $f \overset{sDbx}{\simeq} g$.*

Proof. *Necessity.* Let $\mathcal{P}$ be a complete ring of Darboux functions, such that $f, g \in \mathcal{P}$, and let

$$\mathcal{P}_0 = \{h \in \mathcal{P} : h(0) = 0 \wedge C_h \supset (0,1]\}$$

Then $\mathcal{P}_0$ is a complete ring of Darboux functions, too, and $f, g \in \mathcal{P}_0$. According to the previous theorem, f and g are strongly Darboux homotopic.

Sufficiency. Let $\mathcal{T}$ be a topology finer than the natural topology of the unit interval, such that $I([0,1], \mathcal{T}) = \emptyset$ and $f, g : ([0,1], \mathcal{T}) \to \Re$ are continuous.

Fix $n \in N$. Let V be a $\mathcal{T}$-neigbourhood of 0 such that $f(V) \subset (-\frac{1}{n}, \frac{1}{n}) \supset g(V)$. Then $A_n = V \cap [0, \frac{1}{n})$ is a $\mathcal{T}$-neighbourhood of 0, too. Let $x_n \in A_n \setminus \{0\}$; thus $f(x_n) \in (-\frac{1}{n}, \frac{1}{n}) \ni g(x_n)$. Since f and g are continuous at x_n, there exists $\delta_n > 0$ such that $(x_n - \delta_n, x_n + \delta_n) \subset [0,1)$ and $f((x_n - \delta_n, x_n + \delta_n)) \subset (-\frac{1}{n}, \frac{1}{n}) \supset g((x_n - \delta_n, x_n + \delta_n))$.

Now, we shall define the base: $\mathcal{B} = \{(p,q) \cap [0,1] : p, q \in Q\} \cup \{\{0\} \cup \bigcup_{n=n_0}^{\infty}(x_n - \delta_n, x_n + \delta_n) : n_0 = 1, 2, \ldots\}$.

Let $\mathcal{T}_{\mathcal{B}}$ be the topology generated by the base $\mathcal{B}$ and let $\mathcal{R}$ be the ring of all continuous functions $h : ([0,1], \mathcal{T}_{\mathcal{B}}) \to \Re$. Then $\mathcal{R}$ is a complete ring and

$f, g \in \mathcal{R}$. Observe that all functions from $\mathcal{R}$, considered as functions mapping $[0,1]$ into the real line, are continuous at each point from $(0,1]$ and, consequently, $\mathcal{R} \subset B_1$ (where B_1 denotes the set of all functions of the first class of Baire). Moreover, we may remark that each function from $\mathcal{R}$ fulfills the Young condition ([13]), and so, $\mathcal{R}$ is a complete ring of Darboux functions.

In topology theory, the notion of *homotopy* is connected with the notion of *retract*. So, it seems interesting to consider

the properties of similar notions: *Darboux retract* and *Darboux deformation retract.*

Definition [11]. A subspace B of a space X is said to be a *Darboux retract* of X if there exists a Darboux function - called a Darboux retraction - $r : X \to B$ such that $r(x) = x$ for $x \in B$.
A subset C of X is a *Darboux deformation retract* if C is a Darboux retract and there exists a Darboux homotopy between the identity on X (id_X) and some Darboux retraction $r : X \to C$.

Theorem 4 ([8]) [5] *Let $A \neq \emptyset$ be a subset of a space X, such that $c(A) \leq$ c and $\overline{A} \in G_\delta^*$. Then A is a Darboux retract of X if and only if $A \sqsubset X$ and,*

[4] We do not assume that the functions belonging to $\mathcal{P}$ vanish at 0.

[5] We say that a set A is a set of type

to any $x \in \overline{A} \setminus A$, *we may assign an element* $z_x \in E_A(x)$ *in the way that* $(C \cap A) \cup \{z_x : x \in C \setminus A\}$ *is a connected set for any connected set* $C \subset \overline{A}$.

Problem 3 *The questions connected with the characterization of Darboux deformation retracts are open.*

The following theorems are also connected with the characterization of Darboux retracts and with the questions of the structure of sets which are Darboux retracts.

Theorem 5 ([8]) [6] *Let* X *be a connected* T_6*-space and let* A *be a set of cardinality continuum, such that* $A \in \Xi_X$. *Then* A *is a Darboux retract of* X *if and only if* A *is a closed set.*

Theorem 6 ([8]) *Let* X *be a connected topological space which possesses an open and dense set* V *such that* $X \setminus V$ *is a connected set. Moreover, let the following conditions be satisfied:*

(i) $C \cap V \in \Xi_V$ *for any connected set* $C \subset X$;

(ii) $C \in \Xi_{X \setminus V}$ *for any connected se* $C \subset X \setminus V$;

(iii) if $C \cap V \neq \emptyset \neq C \setminus V$, *then* $X \setminus V \subset \overline{C \cap V}$ *for any connected set* $C \subset X$.

Then each Darboux retract of the space X *is a Borel set.*

Problem 4 ([8]) *It would be interesting to find sufficient conditions (weaker than those in the above theorems) for Darboux retracts to be Borel sets.*

The notions of a Darboux homotopy and Darboux retracts can be applied to the characterization of some topological objects. The following theorem illustrates this fact in the case of a component.

Theorem 7 ([9]) *A subset* C *of a metric separable space* X *is a component of* X *if and only if* C *is a maximal Darboux retract (with respect to the inclusion) such that each Darboux retraction of* C *is weakly Darboux homotopic to each connected function* $f : X \to C$.

F_σ^* ($A \in F_\sigma^*$) if $A = \bigcup_{n=1}^{\infty} F_n$ where F_n are closed sets functionally separated from each closed set disjoint from F_n. We say that $A \in G_\delta^*$ if $X \setminus A \in F_\sigma^*$.

Let $A \subset X$; then the symbol $A \sqsubset X$ means that, for each component C of X, there exists a component C_A of A such that $C_A \subset C$.

The notation $c(A) \leq c$ means that each component of the set A has cardinality not greater than the continuum.

Let $A \subset X$ and $x_0 \in \overline{A}$; then the symbol $E_A(x_0)$ denotes the set of all those $a \in A$ for which:

1° if $x_0 \in \overline{C} \setminus C$, then $a \in \overline{C}$ for each connected set $C \subset A$;

2° if $A \cap C_{x_0} \neq \emptyset$, then $a \in C_{x_0}$ (C_{x_0} denotes the component of X to which x_0 belongs).

[6]Let A and B be any subsets of X. We shall write $A \in \Xi_B$ if, for any distinct points $x, y \in \overline{A}$ of which at least one belongs to B, there exists a connected set $C \subset A$ such that $x \in \overline{C}$ and $y \notin \overline{C}$.

In algebraic topology theory an essential role is played by the notion of *fundamental group*. In particular, lots of questions are connected with the *isomorphism* of the fundamental group (of fixed topological spaces) based at different points. Thanks to this, it is possible to define the notion of the fundamental group of the entire space. Now, we shall give some results connected with this theme.

The fundamental group of the space X based at x_0 will be denoted by $\Pi(X, x_0)$.

Theorem 8 *Let X be a Hausdorff space and let A be a Darboux deformation retract. Then, for each $x \in X \setminus A$, there exists a point $a_x \in A$ such that $\Pi(X, x)$ and $\Pi(X, a_x)$ are isomorphic.*

Theorem 9 *Let X, Y be Hausdorff spaces and let $D(X, Y)$ denote the space of all*

Darboux functions $f : X \to Y$ with the topology of pointwise convergence. Then if $g, h \in D(X, Y)$ are Darboux homotopic, then $\Pi(D(X, Y), g)$ and $\Pi(D(X, Y), h)$ are isomorphic.

Problem 5 *The question connected with the problem similar to that in the above theorem, but with respect to the space $D(X, Y)$ with the compact-open topology or the topology of uniform convergence, seems to be interesting.*

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