Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 786–798

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REGULARITY OF LOCALLY LIPSCHITZ FUNCTIONS ON THE LINE

Abstract

A locally Lipschitz function is regular at points where its lower Dini and Clarke derivatives coincide. For a locally Lipschitz function on an open interval the set of points where the function is regular but not differentiable is at most countable. By constructing a set with unusual metric density properties, and integrating its characteristic function, we produce a locally Lipschitz function which is nowhere regular.

0 Introduction

Since the time of Lebesgue it has been known that there is a strong connection between differentiability properties of locally Lipschitz functions on the line and the metric density behaviour of measurable sets. It is well known that locally Lipschitz functions on the line are differentiable almost everywhere. Regularity (defined below) requires some form of continuity amongst these derivatives. The purpose of this note is to investigate how often a locally Lipschitz function must be regular, and make connections to the possible metric density behaviour of measurable sets.

A function $f : I \to \mathbb{R}$, where I is an open interval, is *locally Lipschitz* if for each $x \in I$ there exist $K, \delta > 0$ such that

$$|f(y) - f(z)| \leq K|y - z| \quad \text{for every } y, z \in I \cap (x - \delta, x + \delta).$$

The *upper* and *lower right Dini derivatives* of f at x are

$$f^+(x) = \limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad f_+(x) = \liminf_{h \to 0+} \frac{f(x+h) - f(x)}{h}.$$

Key Words: differentiability, regularity, locally Lipschitz functions, metric density
Mathematical Reviews subject classification: Primary: 26A24 Secondary: 46G05, 58C20
Received by the editors July 19, 1994

Similarly the *upper* and *lower left Dini derivatives* are

$$f^-(x) = \limsup_{h\to 0^+} \frac{f(x-h)-f(x)}{h} \quad \text{and} \quad f_-(x) = \liminf_{h\to 0^+} \frac{f(x-h)-f(x)}{h}.$$

The function is differentiable from the right at x if $f^+(x) = f_+(x)$, and this common value is denoted $f'_+(x)$. The left derivative occurs when $-f^-(x) = -f_-(x)$ and this value is denoted $f'_-(x)$. The function is *differentiable* when $f'_+(x) = f'_-(x)$, and we simply write $f'(x)$ in this case.

In 1975 Clarke [3] introduced a directional derivative for locally Lipschitz functions on a Banach space. On the line this derivative may be better known as the *strong* or *sharp* derivative. The *right* and *left Clarke derivatives* of f at x are

$$f^o_+(x) = \limsup_{\substack{y\to x \\ h\to 0^+}} \frac{f(y+h)-f(y)}{h} \quad \text{and} \quad f^o_-(x) = \liminf_{\substack{y\to x \\ h\to 0^+}} \frac{f(y+h)-f(y)}{h}$$

and these are finite since the function is locally Lipschitz. Clearly $f^o_-(x) \leq f^o_+(x)$. Also f^o_+ is upper semi-continuous, while f^o_- is lower semi-continuous. We say that f is *regular* at x if $f^o_+(x) = f_+(x)$ and $f^o_-(x) = -f_-(x)$, and that f is *pseudo-regular* if $f^o_+(x) = f^+(x)$ and $f^o_-(x) = -f^-(x)$ [1]. The function is *strictly differentiable* at x if $f^o_-(x) = f^o_+(x)$. If f is regular and differentiable at x then it is strictly differentiable at x.

It is easy to see that pseudo-regularity corresponds to upper semi-continuity of the upper Dini derivatives, and that regularity corresponds to upper semi-continuity of the lower Dini derivatives. Strict differentiability, then, coincides with continuity of all the Dini derivatives. Young and Jurek established that for a locally Lipschitz function f on an open interval I the set of points where f is pseudo-regular is a dense G_δ subset of its domain [9, p151]. One way to see this is to consider [4],

$$f_p^+(x) = \sup_{0<h<\frac{1}{p}} \frac{f(x+h)-f(x)}{h}$$

which is lower semi-continuous. Then $f^+(x) = \lim_{p\to\infty} f_p^+(x)$, so the upper right Dini derivative is the pointwise limit of lower semi-continuous functions, hence is upper semi-continuous at the points of a dense G_δ set. A similar argument on the left completes the proof. However this style of argument fails for regularity, since it expresses the lower Dini derivatives as pointwise limits of upper semi-continuous functions, and there is no reason to expect these to be upper semi-continuous anywhere. A positive result along these lines, however, is that the set of points where the function is differentiable but not strictly differentiable is of the first category. In particular, an everywhere differentiable function is regular at the points of a dense G_δ set, since pseudo-regularity, regularity and strict differentiability coincide for such a function [10].

The evidence above suggests there is no reason to expect a locally Lipschitz function on the line to be regular anywhere. In our second section we recall that the set of points where the function is regular but not differentiable is at most countable. A standard construction (see, for example [6, p97]) immediately yields a function which is regular at no more than countably many points. Perturbing this function can lead to a nowhere regular function. In the third section we look at a particular example of a function defined as the integral of a characteristic function of a measurable set and determine that it has no regular points. This section has the added advantage of constructing a set with abhorrent metric density behaviour — in the terminology introduced there we create a measurable set with no metric boundary.

1 Regular, Non-Differentiable Points are Countable

The following theorem is a special case of a classical result of Young and Levi, [7, p261] .

Theorem 1.1 *Let $f : I \to \mathbb{R}$ be locally Lipschitz on the open interval I, then the set of points where f is regular but not (strictly) differentiable is at most countable.*

We note the implication for a regular function, such as a convex function.

Corollary 1.2 *A regular locally Lipschitz function on an open interval is differentiable at all but a countable set of points.*

Rockafellar [6, p97] considers a locally Lipschitz function defined in the following way. Let E be a measurable subset of $\mathbb{R}$ with the property that both E and E^c have positive measure in every interval. We shall call such a set *ubiquitous*. Then define

$$f(x) = \int_0^x \chi_E(t)\,d\lambda(t)$$

where λ is Lebesgue's measure. It follows from well known results in measure theory [2] that f is locally Lipschitz with $f^\circ_+(x) = 1$ and $f^\circ_-(x) = 0$ for every $x \in \mathbb{R}$. Thus no regular point can be differentiable, and by Theorem 1.1 there are at most countably many regular points.

In order to construct a nowhere regular function it suffices to perturb the function f above to destroy regularity at those points which may possess it. Take a function g which is everywhere regular except at the origin. For example

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Let x_n be an enumeration of the regular points of the function f. Then a nowhere regular function is obtained by perturbing f to

$$\tilde{f}(x) = f(x) + \sum_{n=1}^{\infty} \frac{1}{2^n} g(x - x_n).$$

It would be of interest to know, however, whether it is possible to choose a ubiquitous set E so that the function obtained by integrating its characteristic function is nowhere regular. The following section is dedicated to constructing such a ubiquitous set. Such a set E necessarily has strange metric density properties. Most results about the metric density of measurable sets on the line come modulo a set of measure zero, and are therefore not applicable for our purpose. We will therefore need to develop a construction which allows precise analysis of the metric density behaviour at every point on the line.

2 A Nowhere Regular Function Constructed via a Ubiquitous Set

2.1 Metric Densities

We introduce the following notions as natural generalisations of the concept of metric density studied elsewhere [5]. Given a measurable set $E \subset \mathbb{R}$, the *upper* and *lower right metric densities* of E at x are

$$d^+(E,x) = \limsup_{h \to 0+} \frac{\lambda(E \cap [x, x+h])}{h} \text{ and } d_+(E,x) = \liminf_{h \to 0+} \frac{\lambda(E \cap [x, x+h])}{h}$$

where λ is Lebesgue's measure. When these are equal we refer to the right (metric) density as $d'_+(E,x)$. Similarly on the left we have

$$d^-(E,x) = \limsup_{h \to 0+} \frac{\lambda(E \cap [x-h, x])}{h} \text{ and } d_-(E,x) = \liminf_{h \to 0+} \frac{\lambda(E \cap [x-h, x])}{h}$$

with left density denoted $d'_-(E,x)$. When left and right densities exist and are equal we have the usual metric density. Note however that the definition of *metric density* of E at x is

$$d'(E,x) = \lim_{h\to 0+} \frac{\lambda(E\cap[x-h,x+h])}{2h}$$

so it is possible for the metric density to exist even though the right and left densities are not equal, nor even exist.

We proceed to invent various notions of the measure theoretic interior and boundary of a set in $\mathbb{R}$. The *metric interior* of E is

$$\mathrm{mint}E = \{x \in \mathbb{R} : d'(E,x) = 1\}$$

and the *metric boundary* of E is

$$\mathrm{mbdry}E = \{x \in \mathbb{R} : |d'_+(E,x) - d'_-(E,x)| = 1\}.$$

So x is in the metric interior of E if it is a point of density of E from both right and left, and is in the metric boundary of E if it is a point of density of E from one side, and a point of density of E^c from the other side. Also note that $\mathrm{mbdry}E = \mathrm{mbdry}E^c$, and that some points may not be in the metric interior or boundary of either E or E^c.

We could also consider points which are on the 'edge' of E and E^c in a weaker sense. Consider those x for which $d^+(E,x) = d^+(E^c,x) = 1$, so x is in some sense on the left edge of both E and E^c. If we simultaneously demand $d^-(E,x) = d^-(E^c,x) = 1$ we shall say x is in the *fuzzy metric boundary* of E.

As the notation indicates there is a direct connection between these metric densities and the various derivatives of the function

$$f(x) = \int_0^x \chi_E(t)d\lambda(t)$$

although for 'left' quantities there is the unfortunate appearance of minus signs. Each of the metrically defined sets bares a relationship to the differentiability of the function f. The metric interior of E corresponds to those points where $f'(x) = 1$, and the metric interior of E^c to where $f'(x) = 0$. The metric boundary of E is those points where either f or $-f$ is regular but not differentiable. The fuzzy metric boundary of E is those points where both f and $-f$ are pseudo-regular but not differentiable. It may seem that the fuzzy metric boundary of a set is likely to be small, and of course this is true in measure, however for a ubiquitous set the fuzzy metric boundary is a dense G_δ set [9, p151].

Henceforth we will deal freely in the language of metric densities rather than derivatives since our construction is of the ubiquitous set E rather than the locally Lipschitz function it generates.

2.2 Density Behaviour of Some Cantor Sets

The building blocks of the set we ultimately wish to construct are Cantor sets of a specific type. We study their density properties as a precursor to the horrendous calculation to come. An authoritative account of the construction of Cantor sets and their binary representation may be found in [8].

Consider a Cantor set K_α of measure $(1-\alpha)$ in the interval [0,1] created in the following fashion. Set $K_\alpha^0 = [0,1]$ and create K_α^n by removing the middle open interval of length $\frac{2\alpha}{4^n}$ from each of the 2^{n-1} closed intervals comprising K_α^{n-1}. Now set $K_\alpha = \cap_{n=0}^{\infty} K_\alpha^n$.

Each $k \in K_\alpha$ can be specified in the usual way by a unique binary string k_i, $i \in \mathbb{N}$. We consider the relationship between k and its binary representation k_i. Firstly consider $k \in K_\alpha$ such that $k_i = 0$ for all $i > N$. Then in the set K_α^N there are 2^N closed intervals whose total measure is $(1-\alpha\frac{2^N-1}{2^N})$, of which $\sum_{i=1}^N k_i 2^{N-i}$ lie to the left of k. So the measure of closed intervals to the left of k is

$$\frac{1}{2^N}\left(1-\alpha\frac{2^N-1}{2^N}\right)\left(\sum_{i=1}^{N} k_i 2^{N-i}\right).$$

The measure of open intervals in K_α^N to the left of k is

$$\sum_{i=1}^{N} k_i \frac{2\alpha}{4^i}\left(1+\frac{1}{4}+2\times\frac{1}{4^2}+\dots\dots+2^{N-2}\times\frac{1}{4^{N-1}}\right).$$

Adding the measures of the open and closed intervals to the left of k and simplifying gives us

$$k = \left(1-\alpha\frac{2^N-1}{2^N}\right)\left(\sum_{i=1}^{N}\frac{k_i}{2^i}\right)+\alpha\left(\sum_{i=1}^{N}\frac{k_i}{4^i}\left(2+\frac{2^{N-1}-1}{2^{N-1}}\right)\right).$$

If we allow $N \to \infty$ we obtain a formula valid for any $k \in K_\alpha$, namely

$$k = (1-\alpha)\sum_{i=1}^{\infty}\frac{k_i}{2^i}+3\alpha\sum_{i=1}^{\infty}\frac{k_i}{4^i}.$$

The measure of K_α in the interval [0,k] is given by the first of these two terms.

Lemma 2.1 *The metric density behaviour of the Cantor set* K_α *is*

(i) $\mathrm{mint}K_\alpha^c = K_\alpha^c$

(ii) $\mathrm{mint}K_\alpha = \{k \in K_\alpha : \lim_{i\to\infty} k_i$ *doesn't exist*$\}$

(iii) $\mathrm{mbdry}K_\alpha = \{k \in K_\alpha : \lim_{i\to\infty} k_i = 0\} \cup \{k \in K_\alpha : \lim_{i\to\infty} k_i = 1\}$.

PROOF. The inclusion $K_\alpha^c \subset \text{mint} K_\alpha^c$ is clear from the fact that K_α^c is open. Equality will follow from establishing (ii) and (iii). Consider the right metric density at some $k \in K_\alpha$. If $\lim_{i\to\infty} k_i = 1$ then k is the left endpoint of some open interval in K_α^c, so $d'_+(K_\alpha, k) = 0$. For any other $k \in K_\alpha$ we consider the function

$$\frac{\lambda(K_\alpha \cap [k, k+h])}{h}$$

and note that it is monotone while $k+h$ traverses an open interval in K_α^c. We may therefore suppose that $k + h = m \in K_\alpha$ without sacrificing any of the extremes of the function. If we do this, and let k, m have binary representations k_i, m_i in K_α, then we find

$$\frac{\lambda(K_\alpha \cap [k, m])}{m-k} = \frac{(1-\alpha)(\sum_{i=1}^\infty (m_i - k_i)/2^i)}{(1-\alpha)(\sum_{i=1}^\infty (m_i - k_i)/2^i) + 3\alpha(\sum_{i=1}^\infty (m_i - k_i)/4^i)}$$

which can be written as $\frac{1}{1+R}$ where R is the ratio

$$\frac{3\alpha}{1-\alpha} \frac{\sum_{i=1}^\infty (m_i - k_i)/4^i}{\sum_{i=1}^\infty (m_i - k_i)/2^i}.$$

Now as $m \to k^+$ the early terms of their binary representations agree, so we have $R \to 0$ and therefore $d'_+(K_\alpha, k) = 1$.

Noting that by symmetry the left density at k equals the right density at $1-k$, and that the binary representation of $1-k$ is obtained by changing each digit in the binary representation of k, that is

$$(1-k)_i = k_i + 1 \bmod 2,$$

we see that (ii) and (iii) follow. □

In differentiability terms, this shows that if f is the function obtained by integrating the characteristic function of K_α, then f has derivative zero on K_α^c and derivative one when $\lim_{i\to\infty} k_i$ doesn't exist. It is regular (but not differentiable) when $\lim_{i\to\infty} k_i = 0$, a countable set. When $\lim_{i\to\infty} k_i = 1$ the function $-f$ is regular but not differentiable.

2.3 Constructing a Ubiquitous Set

Recall a measurable set E is ubiquitous if both E and E^c have positive measure in every interval. The general scheme we use to construct such sets is as follows. Begin with a Cantor set of positive measure. Its complement consists of open intervals into each of which we insert a copy of another Cantor set of positive measure, appropriately scaled to the length of the interval. Repeating this process, with care that the resulting set doesn't become too full in measure, yields a ubiquitous set.

More formally, let K_n be a sequence of Cantor sets of type K_α considered in section 2.2 above. (There is nothing particularly special about these sets other than ease of computation.) Set $E_m^m = K_m$ and define

$$E_m^{n+1} = E_m^n \cup \left(\cup_{i=1}^\infty a_{mi}^n + (b_{mi}^n - a_{mi}^n) K_{n+1} \right) \quad (n \geq m)$$

where (a_{mi}^n, b_{mi}^n) is an enumeration of the open intervals comprising $(E_m^n)^c$. Finally set $E_m = \cup_{n=m}^\infty E_m^n$, so E_m is a set created according to the scheme outlined above, commencing with the Cantor set K_m and using the subsequent Cantor set from the sequence at each stage of the construction. Notice that each set E_m contains many scaled copies of the set E_{m+1}. In fact

$$E_m = K_m \cup \left(\cup_{i=1}^\infty a_{mi}^m + (b_{mi}^m - a_{mi}^m) E_{m+1} \right) \qquad (**)$$

and applying this formula recursively we see that

$$E_m \cap [a_{mi}^{m+k}, b_{mi}^{m+k}] = a_{mi}^{m+k} + (b_{mi}^{m+k} - a_{mi}^{m+k}) E_{m+k+1}.$$

That is to say E_m contains shrunk copies of E_n for every $n > m$. Further, for any open interval $J \subset [0,1]$ there exists a k such that $(a_{mi}^{m+k}, b_{mi}^{m+k}) \subset J$ for some i. Therefore E_m is ubiquitous provided $0 < \lambda(E_n) < 1$ for all $n > m$.

Now we concentrate on just one such system of sets. Suppose α_n are the values of the parameter α corresponding to the Cantor sets K_n, and μ_n are the measures of the sets E_1^n. Then

$$\mu_{n+1} = \mu_n + (1 - \mu_n)(1 - \alpha_{n+1}).$$

If we impose $\mu_n = \frac{2^n - 1}{2^n + 1}$, so that the measure of $E_1 = \frac{1}{2}$, we find $\alpha_n = \frac{2^n+1}{2^n+2}$. This gives the measure of K_n to be $\frac{1}{2^n+2}$, and from the recurrence (**) relating the sets E_m to E_{m+1} we have

$$\lambda(E_m) = \frac{1}{2^m + 2} + \frac{2^m + 1}{2^m + 2} \lambda(E_{m+1}).$$

Given that $\lambda(E_1) = \frac{1}{2}$ we compute the measure of E_m to be $\frac{2}{2^m+2}$. Since this is always strictly between 0 and 1 we conclude that all the sets E_m are ubiquitous.

2.4 Binary Addresses for Our Ubiquitous Set

We aim to develop a system of binary addresses for points in $(0,1)$ in such a way that the density behaviour of the ubiquitous set E_1, constructed in section 2.3 above, can be characterised by the asymptotic behaviour of the address. (Compare with Lemma 2.1.)

For any $x \in (0,1)$ let x^n be the largest element of E_1^n less than x. If $x \in E_1$ then x^n is eventually constant and equal to x, otherwise x^n is strictly increasing to x. Setting $x^0 = 0$, for those $x^n > x^{n-1}$ we have

$$x^n \in a_{1j}^{n-1} + (b_{1j}^{n-1} - a_{1j}^{n-1})K_n$$

for some j, and x^n has a unique binary string describing its position within this Cantor set. Let this string be x_i^n. Further, this string does not consist entirely of 1's, since then $x^n = b_{1j}^{n-1}$, which is an element of E_1^{n-1}, and hence $x^{n-1} \geq b_{1j}^{n-1} = x^n$ which is a contradiction. For those n such that $x^n = x^{n-1}$ set $x_i^n = 0$ for all i. In this fashion every $x \in (0,1)$ is described by a unique sequence of binary strings which we'll call its binary address. Note that the sequence x^n becomes constant if and only if $x \in E_1$ and is strictly increasing if and only if $x \in E_1^c$. In the former case the binary strings x_i^n are zero for any sufficiently large n. In the latter case x^n is always the left endpoint of an open interval in $(E_1^n)^c$, and so x_i^n is eventually 1 for each n as $i \to \infty$. We therefore have the following characterisation.

- (i) $x \in E_1$ if and only if $\exists N$ such that $x_i^n = 0$ for all i when $n > N$,
- (ii) $x \in E_1^c$ if and only if $\lim_{i\to\infty} x_i^n = 1$ for every n.

Associated with the binary address of x is a sequence $s_n(x)$ taking values in $\mathbb{N} \cup \{\infty\}$, given by

$$s_n(x) = \sup\{i : x_i^n = 0\}.$$

As noted earlier, given any n there exists some i such that $x_i^n = 0$, so $s_n(x) \geq 1$. We also note at this stage that the left density at x equals the right density at $1-x$ by symmetry, and that the relationship between the binary addresses of x and $1-x$ is given by

$$(1-x)_i^n = \begin{cases} 0 & \text{if } x_j^n = 0 \text{ for all } j\text{, otherwise} \\ (x_i^n + 1) \bmod 2 & \text{if } i < s_n(x) \\ x_i^n & \text{if if } i \geq s_n(x)\ . \end{cases}$$

A consequence of this is that $s_n(x) = s_n(1-x)$ always. For example, if $x = \frac{1}{2}$ then

$$x_i^n = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Applying the rule above to $1-x$ we see that $(1-x)_i^n = x_i^n$, as it should in this case since the binary address of $\frac{1}{2}$ is unique.

It will transpire that the asymptotic behaviour of the sequence $s_n(x)$ is crucial to the density of E_1 at x.

We now establish the relationship between x and its binary address x_i^n. Taking $x^0 = 0$ as before we see that

$$\begin{aligned} x^n - x^{n-1} &= \beta_n(x)\left[(1-\alpha_n)\sum_{i=1}^{\infty}\frac{x_i^n}{2^i} + 3\alpha_n\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}\right] \\ &= \frac{\beta_n(x)}{2^n+2}\left[\sum_{i=1}^{\infty}\frac{x_i^n}{2^i} + 3(2^n+1)\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}\right] \end{aligned}$$

where $\beta_n(x)$ is the length of the interval $(a_{1j}^{n-1}, b_{1j}^{n-1})$ which contains x. Thus $\beta_1(x) = 1$ and $\beta_n(x)$ satisfies

$$\beta_{n+1}(x) = \frac{2\alpha_n}{4^{s_n(x)}}\beta_n(x).$$

So of course if $s_n(x) = \infty$ then $\beta_m(x) = 0$ for all $m > n$. Summing over n gives

$$x = \sum_{n=1}^{\infty}\frac{\beta_n(x)}{2^n+2}\left[\sum_{i=1}^{\infty}\frac{x_i^n}{2^i} + 3(2^n+1)\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}\right]$$

of which the second term represents the measure of open intervals in copies of K_n which are ultimately filled with copies of E_1^{n+1} of measure $\frac{2}{2^{n+1}+2}$. Thus the measure of E_1 in the interval $[0,x]$ is

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\beta_n(x)}{2^n+2}\left[\sum_{i=1}^{\infty}\frac{x_i^n}{2^i} + 3(2^n+1)(\frac{2}{2^{n+1}+2})\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}\right] \\ =&\sum_{n=1}^{\infty}\frac{\beta_n(x)}{2^n+2}\left[\sum_{i=1}^{\infty}\frac{x_i^n}{2^i} + 3\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}\right]. \end{aligned}$$

2.5 Metric Density of Our Ubiquitous Set

It is an easy consequence of Lemma 2.1, and the method of construction of E_1, that every point of E_1 is a point of metric density of E_1. These are precisely those points for which $s_n(x)$ is eventually infinite.

Indeed the asymptotic behaviour of $s_n(x)$ is critical to the density behaviour at x, as the following lemma shows.

Lemma 2.2 *If $x \in (0,1)$ is a point of right metric density of E_1 (E_1^c) then* $\lim_{n\to\infty} s_n(x) - n = \infty$ $(-\infty)$.

PROOF. For any $y \in (x,1)$, with the binary addresses of x, y given by x_i^n, y_i^n, we have

$$\frac{\lambda(E_1 \cap [x,y])}{y-x} = \frac{1}{1+R}$$

where R is given by

$$\frac{\sum_{n=1}^{\infty} 3\frac{2^n}{2^n+2}[\beta_n(y)(\sum_{i=1}^{\infty} y_i^n/4^i) - \beta_n(x)(\sum_{i=1}^{\infty} x_i^n/4^i)]}{\sum_{n=1}^{\infty} \frac{1}{2^n+2}[\beta_n(y)\sum_{i=1}^{\infty} y_i^n(1/2^i+3/4^i) - \beta_n(x)\sum_{i=1}^{\infty} x_i^n(1/2^i+3/4^i)]}.$$

In particular we consider a sequence $y_m \to x^+$ given by

$$(y_m)_i^n = \begin{cases} 1 & \text{if } n = m \text{ and } i = s_m(x) \\ 0 & \text{if } n = m \text{ and } i = s_m(x)+1 \\ x_i^n & \text{otherwise.} \end{cases}$$

So y_m is a number whose binary representation is identical to that of x except in two places, so $s_n(y_m) = s_n(x)$ except when $n = m$, in which case $s_n(y_m) = s_n(x)+1$. We conclude that $\beta_n(y_m) = \beta_n(x)$ for $n \le m$ and $\beta_n(y_m) = \frac{1}{4}\beta_n(x)$ for $n > m$. Calculating R at such points we have

$$\frac{9\frac{2^m}{2^m+2}(\beta_m(x)/4^{s_m(x)+1}) + A}{(2^{s_m(x)+1}+3)\frac{1}{2^m+2}(\beta_m(x)/4^{s_m(x)+1}) + B}.$$

where

$$A = \sum_{n=m+1}^{\infty} 9\frac{2^n}{2^n+2}\beta_n(x)(\sum_{i=1}^{\infty} x_i^n/4^i), \text{ and}$$

$$B = \sum_{n=m+1}^{\infty} \frac{3}{2^n+2}\beta_n(x)(\sum_{i=1}^{\infty} x_i^n(1/2^i+3/4^i))$$

Now we do some estimating, firstly for the summation in the numerator.

$$\begin{aligned}\sum_{n=m+1}^{\infty} \frac{9.2^n}{2^n+2}\beta_n(x)(\sum_{i=1}^{\infty}\frac{x_i^n}{4^i}) &\le \sum_{n=m+1}^{\infty} 3\beta_n(x) \\ &\le 6\beta_{m+1}(x) \\ &\le \frac{1}{2^m+2}\frac{48}{4^{s_m(x)+1}}\beta_m(x)\end{aligned}$$

since $\beta_{n+1}(x) \le \frac{1}{2}\beta_n(x)$ and $\beta_{m+1}(x) = (2\alpha_m\beta_m(x))/4^{s_m(x)}$. We therefore overestimate R by taking the summation in the denominator to be zero, and using our overestimate of the numerator, to get

$$R \le \frac{9.2^m + 48}{2^{s_m(x)+1}+3}.$$

Similarly overestimating the summation in the denominator of R gives

$$\begin{aligned}\sum_{n=m+1}^{\infty}\frac{3}{2^n+2}\beta_n(x)\Big(\sum_{i=1}^{\infty}x_i^n\Big(\frac{1}{2^i}+\frac{3}{4^i}\Big)\Big) &\leq \sum_{n=m+1}^{\infty}3\beta_n(x)\\ &\leq 6\beta_{m+1}(x)\\ &\leq \frac{1}{2^m+2}\frac{48}{4^{s_m(x)+1}}\beta_m(x)\end{aligned}$$

as before. We now underestimate R by taking the summation in the numerator to be zero, and using our overestimate for the denominator. Thus

$$R \geq \frac{9.2^m}{(2^{s_m(x)+1}+3)+48}.$$

Now for x to be a point of right density of E_1 we need $R \to 0$ as $m \to \infty$. From our underestimate of R this requires $s_m(x) - m \to \infty$. Similarly for x to be a point of right density of E_1^c we need $R \to \infty$ as $m \to \infty$. From our overestimate of R this requires $s_m(x) - m \to -\infty$. □

Theorem 2.1 *The metric boundary of E_1 in (0,1) is empty.*

PROOF. If $x \in$ mbdry E_1 then x is a point of right density of either E_1 or E_1^c, and a point of left density of the other. Suppose x is a point of right density of E_1, then $s_n(x) - n \to \infty$ by Lemma 2.2. Also x is a point of left density of E_1^c, so $(1-x)$ is a point of right density of E_1^c, and $s_n(1-x) - n \to -\infty$. However $s_n(x) = s_n(1-x)$, so we have a contradiction. A similar contradiction arises if we suppose x is a point of left density of E_1. □

Corollary 2.1 *The locally Lipschitz function $f : (0,1) \to \mathbb{R}$ given by*

$$f(x) = \int_0^x \chi_{E_1}(t)d\lambda(t)$$

is nowhere regular.

PROOF. No differentiable point of f is regular. The regular, non-differentiable points of f lie in the metric boundary of E_1 which is empty. □

The whole question of allowable metric density behaviour of measurable sets seems to be one which is not well understood. As an example of an open question in this area, must a measurable set on the line (neither full nor zero measure) have at least one point of metric density a half? Of course a point on the metric boundary of a set has metric density a half, but the construction above shows that a set may have no metric boundary.

The author wishes to thank Professor Jonathon Borwein and all at the CECM, Simon Fraser University, for their hospitality and guidance while this work was undertaken, and the referees for their suggested revisions. In particular the observation that a nowhere regular function could be created by beginning with the integral of a ubiquitous set, and perturbing it.

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