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LINEAR SPACES OF DARBOUX DERIVATIVES *

1 Introduction

It is well known that there exist Darboux functions which are not derivatives (see [4], page 97). However, as it was proved in [6], each bounded Darboux function is the "Darboux derivative" (see 2.4 below) of its indefinite integral with respect to a suitable measure and a differentiation system associated with f. The existence of Darboux derivatives which are not Darboux functions was proved in [5] and [9]; moreover, as for Darboux functions, the sum of Darboux derivatives may be not a Darboux derivative.

In this paper we associate to each Darboux derivative f the linear space Δ_f of all the Darboux derivatives with respect to the differentiation system associated with f. We prove that Δ_f contains each function $F \circ f$ where F is continuous on the range of f, and so contains each polynomial in f (see Remarks 4 and 5). Moreover sufficient conditions for a function g to belong to Δ_f are also given (see Propositions 3.8 - 3.10).

We observe that if $f(x) = x$ for each $x \in [a, b]$, then Δ_f is the set of all the Lebesgue integrable functions which are the derivatives of their integrals at every $x \in [a, b]$.

2 Preliminaries

2.1 Measure Associated to a Function

Let $f : [a, b] \to I$ (where I is a bounded subinterval of $\mathbb{R}$) be a real bounded nowhere constant function with $f([a, b]) = I$ and set $\mathcal{F}_f = \{f^{-1}(B) : B \subset$

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I is a Borel set$\}$. It is clear that $\mathcal{F}_f$ is a σ-algebra. Now for each $F \in \mathcal{F}_f$ set $\psi_f(F) = \mu(B)$, where μ is Lebesgue measure on $\mathbb{R}$ and B is a Borel subset of I such that $F = f^{-1}(B)$. It is easy to verify that ψ_f is a finite measure on $\mathcal{F}_f$. Then we can extend it to the complete σ-algebra $\mathcal{M}_f$ of all the ψ_f^* -measurable sets, where ψ_f^* is the outer measure induced by ψ_f on the σ-algebra of all the subsets of $[a,b]$.

Let $\bar{\psi}_f$ denote this extension and call it the *measure associated to f*. It is a finite, complete and completely additive measure on $\mathcal{M}_f$. Moreover $\bar{\psi}(F) = \psi_f(F)$ for each $F \in \mathcal{F}_f$. Denoting outer Lebesgue measure by μ^*, in [7] it was proved that:

i) $\psi_f^*(A) = \mu^*(f(A))$ for each set $A \subset [a,b]$,

ii) if A is ψ_f^*-measurable, then $f(A)$ is μ^*-measurable,

iii) if $A = f^{-1}(T)$ and T is μ^*-measurable, then A is ψ_f^*-measurable.

Moreover, it is well known [2] that to each set $A \subset [a,b]$ there is a set $A^* \in \mathcal{M}_f$ with $A^* \supset A$ and $\psi_f^*(A) = \bar{\psi}_f(A^*)$; A^* is called *a $\bar{\psi}_f$-measure cover of A.*

2.2 Integral with Respect to a Measure

If g is a real bounded $\bar{\psi}_f$-measurable function, it is also $\bar{\psi}_f$-integrable (see [2] page 149); i.e., there exists a unique set function ϕ_f^g on $\mathcal{M}_f$ such that:

(*) ϕ_f^g is completely additive,

(**) if $M \in \mathcal{M}_f$, $h, k \in \mathbb{R}$ and $\forall x \in M$ we have $h \le g(x) \le k$, then

$$h \cdot \bar{\psi}_f(M) \le \phi_f^g(M) \le k \cdot \bar{\psi}_f(M).$$

We call $\phi_f^g(M)$ *the integral of g on M with respect to $\bar{\psi}_f$* and set $\phi_f^g(M) = \int_M g\, d\bar{\psi}_f$. Extend ϕ_f^g to each subset A of $[a,b]$ as follows: $\phi_f^{*g}(A) = \phi_f^g(A^*)$, where A^* is a $\bar{\psi}_f$-measure cover of A. If $g = f$, put $\phi_f^f = \phi_f$ and $\phi_f^{*f} = \phi_f^*$.

2.3 Darboux Systems

For any $x_1, x_2 \in [a,b]$ ($x_1 \ne x_2$), denote by $I_{(x_1,x_2)}$ the closed interval $[x_1,x_2]$ if $x_1 < x_2$, and the closed interval $[x_2,x_1]$ otherwise. If $f(x_1) \ne f(x_2)$, then set $Q_{x_1x_2}^f = f^{-1}(I_{(f(x_1),f(x_2))}) \cap I_{(x_1,x_2)}$. The system $Q^f = \{Q_{x_1x_2}^f : x_1, x_2 \in [a,b]$ and $f(x_1) \ne f(x_2)\}$ is called the system associated with f. Given a sequence $\{Q_{x_n'x_n''}^f\}$ we say that it converges to x_0 in the Darboux sense with respect to f, and we write $Q_{x_n'x_n''}^f \xrightarrow{\mathcal{D}} x_0$, when $x_n' \to x_0, x_n'' \to x_0$ and $f(x_n') \to$

$f(x_0), f(x''_n) \to f(x_0)$. We say that the system $\mathcal{Q}^f$ is the Darboux system associated to f when for each $x \in [a,b]$ there is a sequence $\{Q^f_{x'_n x''_n}\}$ converging to x in the Darboux sense with respect to f and $\psi^*_f(Q^f_{x'_n x''_n}) \neq 0$ for each $n \in \mathbb{N}$.

2.4 Darboux Derivatives

A real bounded function g defined on $[a,b]$ will be called a Darboux derivative if there exists a real bounded nowhere constant function f on $[a,b]$, a σ-algebra $\mathcal{E}$ of subsets of $[a,b]$ and two set functions ϕ and ψ on $\mathcal{E}$ such that:

α) the system $\mathcal{Q}^f$ associated with f is a Darboux-system and $\mathcal{Q}^f \subset \mathcal{E}$,

β) for each $x \in [a,b]$, there exist a sequence $\{Q^f_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}^f$ with $Q^f_n \xrightarrow{\mathcal{D}} x$ and $\psi(Q^f_n) \neq 0$, for each $n \in \mathbb{N}$,

γ) $g(x) = \lim\limits_{Q^f_n \xrightarrow{\mathcal{D}} x} \dfrac{\phi(Q^f_n)}{\psi(Q^f_n)} = D_{\mathcal{D}}(x, \phi, \psi, \mathcal{Q}^f)$ for each $x \in [a,b]$ and for each $Q^f_n \xrightarrow{\mathcal{D}} x$ with $\psi(Q^f_n) \neq 0$.

The class of all Darboux derivatives will be denoted by $\Delta_{\mathcal{D}}$. It is not empty since each bounded nowhere constant Darboux function f is the Darboux derivative of its $\bar{\psi}_f$-integral with respect to the measure $\bar{\psi}_f$ and the system $\mathcal{Q}^f$ associated to f (see [6]).

Moreover, let f be a bounded nowhere constant function on $[a,b]$ and for each x_1, x_2 with $f(x_1) \neq f(x_2)$ set

$$A^f_{x_1x_2} = \{y \in I_{(f(x_1), f(x_2))};\, f(x) \neq y \text{ for each } x \in I_{(x_1,x_2)}\}.$$

Denote by $\mathcal{Q}_{\mathcal{D}}$ the family of all bounded nowhere constant functions f such that $\mu(A^f_{x_1x_2}) = 0$ for each x_1, $x_2 \in [a,b]$, and by $\mathcal{A}$ the family of all bounded nowhere constant functions f such that, for all $x_1, x_2 \in [a,b]$, $\mu_*(A^f_{x_1x_2}) = 0$, where μ_* is Lebesgue inner measure. It is known (see [5] and [9]) that $\mathcal{D} \subset \mathcal{Q}_{\mathcal{D}} \subset \mathcal{A} \subset \Delta_{\mathcal{D}}$, where $\mathcal{D}$ is the class of all real bounded nowhere constant Darboux functions defined on $[a,b]$.

3 Main Results

3.1

Here we define a real linear space whose elements are Darboux derivatives with respect to the system $\mathcal{Q}^f$ and with respect to the measure $\bar{\psi}_f$ associated with a given function $f \in \mathcal{D}$.

Proposition 3.1 *Let $f : [a,b] \to [l,L]$ be a nowhere constant Darboux function such that $f([a,b]) = [l,L]$ and let $\mathcal{X} = \{F \circ f; F \in C[l,L]\}$ where $C[l,L]$ denotes the set of all continuous functions on $[l,L]$. Then:*

i) $g \in \mathcal{X} \Longrightarrow g$ is a Darboux function,

ii) $\mathcal{X}$ is a real linear space,

iii) $g \in \mathcal{X} \Longrightarrow g$ is $\bar{\psi}_f$-measurable,

iv) $\int_A g\, d\bar{\psi}_f = \int_{f(A)} F(y)\, d\mu$, for all $A \in \mathcal{M}_f$ and $g = F \circ f \in \mathcal{X}$.

PROOF.

i) Let $z_0 \in \mathbb{R}$ and let $x_1, x_2 \in [a,b]$ be such that $(F \circ f)(x_1) < (F \circ f)(x_2)$ and $(F \circ f)(x_1) < z_0 < (F \circ f)(x_2)$. Since F is continuous, there exists $y_0 \in I_{(f(x_1),f(x_2))}$ such that $F(y_0) = z_0$. Since f is a Darboux function, there exists $x_0 \in I_{(x_1,x_2)}$ such that $f(x_0) = y_0$. It follows that $(F \circ f)(x_0) = F(f(x_0)) = F(y_0) = z_0$, i.e. $F \circ f \in \mathcal{D}$.

ii) Let $(F_1 \circ f)$, $(F_2 \circ f) \in \mathcal{X}$. Then $(F_1 \circ f) + (F_2 \circ f) \equiv (F_1 + F_2) \circ f \in \mathcal{X}$ and $K \cdot (F \circ f) \equiv (K \cdot F) \circ f \in \mathcal{X}$ for all $K \in \mathbb{R}$.

iii) To prove the $\bar{\psi}_f$-measurability of $F \circ f$, we observe that

$$(F \circ f)^{-1}(I_{((F\circ f)(x_1),(F\circ f)(x_2))}) = f^{-1}[F^{-1}(I_{(F(f(x_1)),F(f(x_2)))})];$$

moreover $F^{-1}(I_{(F(f(x_1)),F(f(x_2)))})$ is a Borel set, because F, being continuous, is Borel measurable. Finally $f^{-1}[F^{-1}(I_{(F(f(x_1)),F(f(x_2)))})]$ is $\bar{\psi}_f$-measurable since f is $\bar{\psi}_f$-measurable.

iv) The boundedness and the $\bar{\psi}_f$-measurability of g imply its $\bar{\psi}_f$-integrability; moreover the second integral is a completely additive set function on $\mathcal{M}_f$. In fact, if $\{A_n\}$ is a sequence of sets in $\mathcal{M}_f$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\{f(A_n)\}$ is a sequence of Lebesgue measurable sets and $f(A_i) \cap f(A_j)$ is a μ-nullset for $i \neq j$, since, for every i there are a Borel set B_i and a $\bar{\psi}_f$-nullset N_i such that $A_i = f^{-1}(B_i) \cup N_i$ and $f^{-1}(B_i) \cap N_i = \emptyset$. Then

$$\int_{f(\cup_n A_n)} F(y)\, d\mu = \int_{\cup_n f(A_n)} F(y)\, d\mu = \sum_{n \in N} \int_{f(A_n)} F(y)\, d\mu.$$

Moreover if $A \in \mathcal{M}_f$ and if there are $h, k \in \mathbb{R}$ such that $h \leq (F \circ f)(x) \leq k$ for every $x \in A$, then $h \leq F(y) \leq k$ for every $y \in f(A)$, and by the μ-integrability of $F(y)$ it follows that $h \cdot \mu(f(A)) \leq \int_{f(A)} F(y)\, d\mu \leq k \cdot \mu(f(A))$. Then the claim follows by the equality $\mu(f(A)) = \bar{\psi}_f(A)$ and by the definition of $\bar{\psi}_f$-integral. $\square$

Proposition 3.2 *For all $F \in C[l,L]$ let $\phi_f^{F\circ f}(A) = \int_A (F\circ f)\, d\bar{\psi}_f$ (see 2.2). Then for each $x \in [a,b]$ we get $(F\circ f)(x) = D_{\mathcal{D}}(x, \phi_f^{*F\circ f}, \psi_f^*, Q^f)$.*

PROOF. Let $x_0 \in [a,b]$. If $Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x_0$ with respect to f, then $Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x_0$ also with respect to $F\circ f$. Now put $Q_n^{*f} = f^{-1}(I_{(f(x'_n),f(x''_n))})$. For all $x \in Q_n^{*f}$

$$i_n = \inf_{y\in I_{(f(x'_n),f(x''_n))}} F(y) \le (F\circ f)(x) \le \sup_{y\in I_{(f(x'_n),f(x''_n))}} F(y) = s_n$$

and, using the $\bar{\psi}_f$-integrability of $F\circ f$, it follows that $i_n \le \dfrac{\phi_f{}^{F\circ f}(Q_n^{*f})}{\bar{\psi}_f(Q_n^{*f})} \le s_n$. Moreover, observing that for $n\to\infty$, $\{i_n\}$ and $\{s_n\}$ converge to $(F\circ f)(x_0)$ and that Q_n^{*f} is a $\bar{\psi}_f$-measure cover of $Q^f_{x'_n x''_n}$ (see 2.1), we obtain

$$\begin{aligned}(F\circ f)(x_0) &= \lim_{n\to\infty} \frac{\phi_f^{F\circ f}(Q_n^{*f})}{\bar{\psi}_f(Q_n^{*f})} = \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x_0} \frac{\phi_f^{*F\circ f}(Q^f_{x'_n x''_n})}{\psi_f^*(Q^f_{x'_n x''_n})} \\ &= D_{\mathcal{D}}(x_0, \phi_f^{*F\circ f}, \psi_f^*, Q^f).\end{aligned}$$

□

Remark 1 The properties ii), iii), iv) and the conclusion of Proposition 3.2 are also true if f is the Darboux derivative of its $\bar{\psi}_f$-integral with respect to $\bar{\psi}_f$ and the system Q^f.

Remark 2 If $F\circ f$ is a nowhere constant function on $[l,L]$, from condition i) of Proposition 3.1 (see [6]), it follows that $(F\circ f)(x) = D_{\mathcal{D}}(x, \phi^*_{F\circ f}, \psi^*_{F\circ f}, Q^{F\circ f})$ for each $x \in [a,b]$. Indeed the above derivative depends on the system and on the measure associated to the function $F\circ f$.

3.2

Given a real bounded nowhere constant function f on $[a,b]$ with connected range, set

$$\begin{aligned}\Delta_f = \{g : [a,b] \to \mathbb{R};\ & g \text{ is bounded, } \bar{\psi}_f\text{-measurable and} \\ & g(x) = D_{\mathcal{D}}(x, \phi_f^{*g}, \psi_f^*, Q^f)\ \forall x \in [a,b]\},\end{aligned}$$

where $\phi_f^{*g}(A) = \int_A g\, d\psi_f^*\ \forall A \subset [a,b]$ (see 2.2). If $f \in \Delta_f$, we prove that Δ_f is a linear space and we give some sufficient conditions in order that a bounded function g be the derivative of its $\bar{\psi}_f$-integral with respect to $\bar{\psi}_f$ and the system Q^f.

3.2.1

Let $\mathbf{C}$ be the class of all real bounded functions defined on $[a, b]$ and having connected range.

Proposition 3.3 *Let $f, g \in \mathbf{C}$ and suppose that f is $\bar{\psi}_g$-measurable. Then for each $A \subset [a, b]$ there is a set $A^* \in \mathcal{M}_g$, which is a $\bar{\psi}_f$-measure cover of A.*

PROOF. For each A there is a Borel subset $B_f \subset f([a, b])$ such that $A^* = f^{-1}(B_f)$ is a $\bar{\psi}_f$-measure cover of A (see [2]). Then A^* is $\bar{\psi}_g$-measurable. □

Proposition 3.4 *Given f, $g \in \mathbf{C}$ such that f is $\bar{\psi}_g$-measurable and g is $\bar{\psi}_f$-measurable, for each set $A \subset [a, b]$ there exists a set A^* which is a measure cover of A with respect to $\bar{\psi}_f$ and $\bar{\psi}_g$.*

PROOF. Let $A \subset [a, b]$. By Proposition 3.3 there exist A_1^* in $\mathcal{M}_g$ which is a $\bar{\psi}_f$-measure cover of A and A_2^* in $\mathcal{M}_f$ which is a $\bar{\psi}_g$-measure cover of A. Then the set $A^* = A_1^* \cap A_2^*$ is the desired set. In fact $A \subset A^*$ and $\psi_f^*(A) \leq \bar{\psi}_f A^*) \leq \bar{\psi}_f(A_1^*)$ and $\psi_g^*(A) \leq \bar{\psi}_g(A^*) \leq \bar{\psi}_g(A_2^*)$. Then from the properties of A_1^* and of A_2^* it follows that $\psi_f^*(A) = \bar{\psi}_f(A^*)$ and $\psi_g^*(A) = \bar{\psi}_g(A^*)$. □

Proposition 3.5 *Let f, $g \in \mathbf{C}$ with f $\bar{\psi}_g$-measurable and let $\{A_n\}$ be a sequence of sets such that $A_n \in \mathcal{M}_f \cap \mathcal{M}_g$ with $\bar{\psi}_f(A_n) > 0$ and $\bar{\psi}_g(A_n) > 0$ for each n. Suppose that for each n there exist $h_n, k_n \in \mathbb{R}$ such that $f(A_n) \subseteq [h_n, k_n]$ and $\lim_{n\to\infty} (k_n - h_n) = 0$. Then*

$$\lim_{n\to\infty} \bar{\psi}_f(A_n) = 0 \text{ and } \lim_{n\to\infty} \left[\frac{\phi_g{}^f(A_n)}{\bar{\psi}_g(A_n)} - \frac{\phi_f(A_n)}{\bar{\psi}_f(A_n)}\right] = 0.$$

PROOF. $\bar{\psi}_f(A_n) = \mu(f(A_n)) \leq k_n - h_n$. Moreover, as f is $\bar{\psi}_g$-measurable and bounded, it is $\bar{\psi}_g$-integrable on every A_n. Then

$$h_n\bar{\psi}_g(A_n) \leq \phi_g^f(A_n) \leq k_n\bar{\psi}_g(A_n). \tag{1}$$

Since f is $\bar{\psi}_f$-integrable,

$$h_n\bar{\psi}_f(A_n) \leq \phi_f(A_n) \leq k_n\bar{\psi}_f(A_n). \tag{2}$$

Now dividing (1) by $\bar{\psi}_g(A_n)$ and (2) by $\bar{\psi}_f(A_n)$, we get $h_n \leq \dfrac{\phi_g^f(A_n)}{\bar{\psi}_g(A_n)} \leq k_n$ and $h_n \leq \dfrac{\phi_f(A_n)}{\bar{\psi}_f(A_n)} \leq k_n$. Hence, $0 \leq \left|\dfrac{\phi_g^f(A_n)}{\bar{\psi}_g(A_n)} - \dfrac{\phi_f(A_n)}{\bar{\psi}_f(A_n)}\right| \leq k_n - h_n$. □

Proposition 3.6 *If $f \in \Delta_f$ is $\bar{\psi}_g$-measurable, $g \in \mathbf{C}$ is $\bar{\psi}_f$-measurable and for each $x \in [a,b]$ there exists a sequence $\{Q^f_n\} \subset \mathcal{Q}^f$ such that $Q^f_n \xrightarrow{\mathcal{D}} x$ with $\psi^*{}_g(Q^f_n) \neq 0$, then $f(x) = D_{\mathcal{D}}(x, \phi^*{}^f_g, \psi^*{}_g, \mathcal{Q}^f)$ for each $x \in [a,b]$.*

PROOF. Let $\{Q^f_{x'_n x''_n}\}$ be a sequence belonging to $\mathcal{Q}^f$ and converging to x in the Darboux sense with respect to f with $\psi^*{}_g(Q^f_{x'_n x''_n}) \neq 0$. Moreover let $Q^*{}_n$ be a measure cover of $Q^f_{x'_n x''_n}$ with respect to $\bar{\psi}_f$ and $\bar{\psi}_g$ such that $Q^*{}_n \subseteq f^{-1}(I_{(f(x'_n),f(x''_n))})$. By Propositions 3.4 and 3.5

$$f(x) = \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*{}_f(Q^f_{x'_n x''_n})}{\psi^*{}_f(Q^f_{x'_n x''_n})} = \lim_{n\to\infty} \frac{\phi_f(Q^*{}_n)}{\bar{\psi}_f(Q^*{}_n)} = \lim_{n\to\infty} \frac{\phi^f_g(Q^*{}_n)}{\bar{\psi}_g(Q^*{}_n)}$$

$$= \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^{*f}_g(Q^f_{x'_n x''_n})}{\psi^*{}_g(Q^f_{x'_n x''_n})} = D_{\mathcal{D}}(x, \phi^*{}^f_g, \psi^*_g, \mathcal{Q}^f).$$

3.2.2

Given $f \in \Delta_f$ we prove the following assertion.

Proposition 3.7 *Δ_f is a real linear space.*

PROOF. Let $f_1, f_2 \in \Delta_f$. For each $x \in [a,b]$ and for each sequence $\{Q^f_{x'_n x''_n}\}$ converging to x in the Darboux sense with respect to f

$$f_1(x) + f_2(x) = \lim_{Q^f_{x'_n, x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*{}^{f_1}_f(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n, x''_n})} + \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*{}^{f_2}_f(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})}$$

$$= \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*_f{}^{(f_1+f_2)}(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})} = D_{\mathcal{D}}(x, \phi^*_f{}^{(f_1+f_2)}, \psi^*_f, \mathcal{Q}^f).$$

Moreover for each $k \in \mathbb{R}$ and $f_1 \in \Delta_f$

$$(k \cdot f_1)(x) = k \cdot f_1(x) = k \cdot \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*_f{}^{f_1}(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})}$$

$$= \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*_f{}^{(kf_1)}(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})} = D_{\mathcal{D}}(x, \phi^*_f{}^{(kf_1)}, \psi^*_f, \mathcal{Q}^f).$$

□

Proposition 3.8 *Let $f \in \Delta_f$ and suppose that:*

1) g is a bounded, $\bar{\psi}_f$-measurable function defined on $[a,b]$,

2) for sequences $\{x'_n\}$ and $\{x''_n\}$ converging to x such that $f(x'_n) \neq f(x''_n)$ and $\{f(x'_n)\}$, $\{f(x''_n)\}$ converge to $f(x)$ there exist two sequences $\{\bar{x}'_n\}$ and $\{\bar{x}''_n\}$ converging to x such that $g(\bar{x}'_n) \neq g(\bar{x}''_n)$ and $\{g(\bar{x}'_n)\}, \{g(\bar{x}''_n)\}$ converge to $g(x)$, with $g(f^{-1}(I_{(f(x'_n),f(x''_n))})) \subset I_{(g(\bar{x}'_n),g(\bar{x}''_n))}$.

Then $g \in \Delta_f$.

PROOF. Let $\{Q^f_{x'_n x''_n}\} \subset \mathcal{Q}^f$ be a sequence converging to x in the Darboux sense with respect to f and $\psi^*_f(Q^f_{x'_n x''_n}) \neq 0$ for each $n \in \mathbb{N}$. If the set $f^{-1}(I_{(f(x'_n),f(x''_n))})$ is not a $\bar{\psi}_f$-measure cover of $Q^f_{x'_n x''_n}$, then there exists such a set Q^{*f}_n contained in $f^{-1}(I_{(f(x'_n),f(x''_n))})$. Consequently, without loss of generality, for each $t \in Q^{*f}_n$ we can write $g(\bar{x}'_n) \leq g(t) \leq g(\bar{x}''_n)$ and, by the $\bar{\psi}_f$-integrability of g, it follows that $g(\bar{x}'_n)\bar{\psi}_f(Q^{*f}_n) \leq \phi^g_f(Q^{*f}_n) \leq g(\bar{x}''_n)\bar{\psi}_f(Q^{*f}_n)$. Hence, $g(\bar{x}'_n) \leq \dfrac{\phi^g_f(Q^{*f}_n)}{\bar{\psi}_f(Q^{*f}_n)} \leq g(\bar{x}''_n)$, and

$$\lim_{n\to\infty} \frac{\phi^g_f(Q^{*f}_n)}{\bar{\psi}_f(Q^{*f}_n)} = g(x) = D_{\mathcal{D}}(x, \phi^{*g}_f, \psi^*{}_f, \mathcal{Q}^f).$$

□

Proposition 3.9 *If $f \in \Delta_f, g \in \mathbf{C}$ satisfy the conditions of Propositions 3.6 and 3.8, then for each $x \in [a,b]$ we have $g(x) = D_{\mathcal{D}}(x, \phi^*_g, \psi^*_g, \mathcal{Q}^f)$.*

PROOF. Let $\{Q^f_{x'_n x''_n}\}$ be a sequence converging to x in the Darboux sense with respect to f. By Proposition 3.4 there exists, for each n, a set $Q^{*f}_n \subset f^{-1}(I_{(f(x'_n),f(x''_n))})$ which is a measure cover of $Q^f_{x'_n x''_n}$ with respect to $\bar{\psi}_f$ and $\bar{\psi}_g$, such that $\dfrac{\phi^{*g}_f(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})} = \dfrac{\phi^g_f(Q^{*f}_n)}{\bar{\psi}_f(Q^{*f}_n)}$ and $\dfrac{\phi_g(Q^{*f}_n)}{\bar{\psi}_g(Q^{*f}_n)} = \dfrac{\phi^*_g(Q^f_{x'_n x''_n})}{\psi^*_g(Q^f_{x'_n x''_n})}$. Then by Proposition 3.5,

$$g(x) = \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^{*g}_f(Q^f_{x'_n x'_n})}{\psi^*_f(Q^f_{x'_n x''_n})} = \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*_g(Q^f_{x'_n x''_n})}{\psi^*{}_g(Q^f_{x'_n x''_n})}.$$

□

Proposition 3.10 *Let $f \in \Delta_f$, $g \in \Delta_g$ and suppose that:*

1) g is $\bar{\psi}_f$-measurable,

2) for each $Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x$ with respect to f, there is $Q^g_{\bar{x}'_n \bar{x}''_n} \xrightarrow{\mathcal{D}} x$ with respect to g such that there exists, for each n, a $\bar{\psi}_g$-measure cover Q^*_n of $Q^g_{\bar{x}'_n \bar{x}''_n}$ and $\bar{\psi}_f$-measure cover of $Q^f_{x'_n x''_n}$.

Then $g \in \Delta_f$.

PROOF. By conditions 1) and 2) and by Proposition 3.5 it follows that

$$g(x) = \lim_{Q^g_{\bar{x}'_n \bar{x}''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^*_g(Q^g_{\bar{x}'_n \bar{x}''_n})}{\psi^*_g(Q^g_{\bar{x}'_n \bar{x}''_n})} = \lim_{n\to\infty} \frac{\phi_g(Q^*{}_n)}{\bar{\psi}_g(Q^*{}_n)} = \lim_{n\to\infty} \frac{\phi^g_f(Q^*_n)}{\bar{\psi}_f(Q^*_n)}$$

$$= \lim_{Q^f_{x'_n x''_n} \xrightarrow{\mathcal{D}} x} \frac{\phi^{*g}_f(Q^f_{x'_n x''_n})}{\psi^*_f(Q^f_{x'_n x''_n})} = D_{\mathcal{D}}(x, \phi^{*g}_f, \psi^*_f, Q^f).$$

□

Remark 3 If $f(x) = x$ for all $x \in [a,b]$, then Δ_f contains only the bounded and μ^*-measurable functions $g(x)$ which are the derivatives of their μ-integrals at each $x \in [a,b]$ (see [1], pages 37–38).

Remark 4 If $f \in \Delta_f$, $f \geq 0$, then Δ_f contains the ring

$$\mathcal{H} = \{f_n(x) = \sum_{i=1}^{n} a_i f^i(x); a_i \in \mathbb{R}, n \in \mathbb{N}\},$$

where $f^i(x) = (f(x))^i$.

PROOF. Since Δ_f is a linear space, it is enough to show that $f^i(x)$ is in Δ_f. In fact, for each x_1, $x_2 \in [a,b]$ we have $(f^i)^{-1}(I_{(f^i(x_1), f^i(x_2))}) = f^{-1}(I_{(f(x_1), f(x_2))})$. Then $f^i(x)$ is $\bar{\psi}_f$-measurable and $Q^{f^i}_{x_1 x_2} = Q^f_{x_1 x_2}$ for each x_1, $x_2 \in [a,b]$. Moreover, if $\{Q^f_{x'_n x''_n}\}$ is a sequence converging in the Darboux sense with respect to f, then it is a sequence converging in the Darboux sense with respect to f^i and the conditions of Proposition 3.9 are verified. □

Remark 5 If $f \in \Delta_f$, $g \in \Delta_g$ and if $g \geq 0$ satisfies the conditions of Proposition 3.8 (or the conditions of Proposition 3.10), then the ring

$$\mathcal{G} = \{g_n(x) = \sum_{k=1}^{n} b_k g^k(x); b_k \in \mathbb{R}, n \in \mathbb{N}\}$$

is contained in Δ_f.

PROOF. For each $x_1, x_2 \in [a,b]$ and $k \in \mathbb{N}$ we have $(g^k)^{-1}(I_{(g^k(x_1),g^k(x_2))}) = g^{-1}(I_{(g(x_1),g(x_2))})$. Then g^k is $\bar{\psi}_f$-measurable and $Q^{g^k}_{x_1x_2} = Q^{g}_{x_1x_2}$ for each x_1, $x_2 \in [a,b]$. Moreover $Q^{g^k}_{x'_n x''_n} \xrightarrow{\mathcal{D}} x$ with respect to g^k when $Q^{g}_{x'_n x''_n} \xrightarrow{\mathcal{D}} x$ with respect to g. Finally condition 2) of Proposition 3.10 (or condition 2) of Proposition 3.8) is verified for g^k. Hence $g^k \in \Delta_f$.

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