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POINTS OF UNIFORM CONVERGENCE AND OSCILLATION OF SEQUENCES OF FUNCTIONS*

Abstract

Various types of local convergence of sequences of functions are investigated in this paper. This investigation is based on the concept of oscillation of sequence of function. Applications of proved results to differentiability of real functions are given.

1 Introduction

It is known that the pointwise limit of a sequence of functions does not carry many important properties of the members of the sequence onto the limit function. Consequently, various types of convergence have been introduced which are stronger than the pointwise convergence. In the first part we study properties of some local types of uniform convergence which are sufficient for proving the continuity at one point of pointwise limit of a sequence of functions. In the second part we discuss the topological structure of the sets of all points of a local type uniform convergence. In the third part we give some applications of our results. We give a new proof of the well-known theorem according to which the set of discontinuity points of an arbitrary function in the first Baire class is a set of the first Baire category. We prove that the set of strong differentiability points of any continuous function is a G_δ set. Applying this result we obtain new proofs of the known facts that the set of all strong (or uniform symmetric) differentiability points of continuous symmetric differentiable function is residual.

Key Words: Locally uniform convergence, uniform and quasiuniform convergence at a point, oscillation of sequence of functions, strong differentiability, uniform differentiability, symmetric differentiability, residual set

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2 Localizations of uniform and quasiuniform convergence

We consider a sequence of functions $(f_n : X \to Y)_{n\geq 1}$, where X is a topological space, and Y is a metric space equipped with a metric ϱ. Let

$$K := \{x \in X : \ (f_n(x))_{n\geq 1} \text{ converges in } Y\}$$

be the domain of convergence of the sequence $(f_n)_{n\geq 1}$ and let the function $f : K \longrightarrow Y$ be the pointwise limit of this sequence i.e.

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in K).$$

If the sequence $(f_n)_{n\geq 1}$ converges uniformly on a set $M \subset K$ to the function f, we shall write $f_n \rightrightarrows f$ *on* M.

A type of convergence weaker than the uniform convergence (see [1] p.739) is **locally uniform convergence on a set** $M \subset K$ which means, that

$$\forall a \in M \ \exists \text{ a neighbourhood } O(a) \ : \ f_n \rightrightarrows f \text{ on } O(a) \cap M.$$

If a sequence $(f_n)_{n\geq 1}$ converges uniformly on a set M to the function f, then it converges locally uniformly on M to the same function f. The converse is not true. For example the sequence $(x^n)_{n\geq 1}$ converges locally uniformly on the set $M = (0, 1)$ but it is not uniformly convergent on M.

The next definition introduces the local form of uniform convergence (see [6] p.589) and, we shall show that it is sufficient to guarantee the continuity of limit function at one point.

Definition 1 *A sequence of functions* $(f_n : X \to Y)_{n\geq 1}$ *is said to* **converge locally uniformly at a point** $a \in K$ **(to the function** f**)** if there is a neighbourhood $O(a)$ of the point a such that

$$f_n \rightrightarrows f \text{ on } O(a) \cap K.$$

It is evident that $(f_n)_{n\geq 1}$ converges locally uniformly at each point of the domain of convergence K iff $(f_n)_{n\geq 1}$ converges locally uniformly on the set K. For a set $M \subset K, M \neq K$ the implication "$\Leftarrow$" may fail. This happens for the sequence $(x^n)_{n\geq 1}$ which is convergent on the set $K = (-1, 1]$ and is locally uniformly convergent on the set $M = \{1\}$ but does not converge locally uniformly at 1. The locally uniform convergence on a set M implies the locally uniform convergence at some topologically distinguished points of the set M.

Proposition 1 *Let a sequence* $(f_n)_{n\geq 1}$ *converge locally uniformly on a set* M *and* $a \in M$ *be an interior point of* M *relative to the domain of convergence* K*. Then* $(f_n)_{n\geq 1}$ *converges locally uniformly at the point* a.

PROOF. By the hypothesis of the locally uniform convergence on the set M of $(f_n)_{n\geq 1}$ to the function f, there is a neighbourhood $O(a)$ such that

$$f_n \rightrightarrows f \text{ on } O(a) \cap M.$$

As a is an interior point of M relative to K, there is a neighbourhood $V(a)$ of a such that $V(a) \cap K \subset M$. For the neighbourhood $W(a) := O(a) \cap V(a)$ we have $W(a) \cap K \subset O(a) \cap M \subset K$ and so

$$f_n \rightrightarrows f \text{ on } W(a) \cap K,$$

and this means that $(f_n)_{n\geq 1}$ converges locally uniformly at the point a. □

Let us denote $LU(f_n)_{n\geq 1}$ the set of all points of locally uniform convergence of the sequence $(f_n)_{n\geq 1}$. If we denote $int_K M$ the relative interior of a set $M \subset K$ relative to the domain of convergence K, we can state Proposition 2 in the form

$$int_K M \subset LU(f_n)_{n\geq 1}$$

for each subset M of K such that the sequence $(f_n)_{n\geq 1}$ converges locally uniformly on the set M.

It is easy to verify that if a sequence f_n $(n \geq 1)$ of functions continuous at a point a converges locally uniformly at the point a to the function f, then f is continuous at a.

The continuity of limit function of a sequence of functions continuous at a point can be guaranteed even by a weaker form of locally uniform convergence. To show this we introduce the following Definition 2 due to C. Goffman (for real functions defined on $\mathbb{R}$, cf. [3] p.149).

Definition 2 *We say that a* **sequence** $(f_n : X \to Y)_{n\geq 1}$ **converges uniformly at a point** $a \in X$ if

(1) $\forall \varepsilon > 0\ \exists O(a)\ \exists n_0 \in \mathbb{N}\ \forall m, n > n_0\ \forall x \in O(a) :\ \varrho(f_n(x), f_m(x)) < \varepsilon.$

Remark 1 *(a) In the previous definition, the convergence of the sequence $(f_n)_{n\geq 1}$ is not assumed. Therefore it would be better to use the term "$(f_n)_{n\geq 1}$ is* **uniformly Cauchy sequence at a point** *a." In what follows we shall consider mainly the sequences which converge on their definition domain, so we can use the Goffman's term.*

(b) In [6] p.589 the author introduced a little different definition of uniform convergence at a point by the following condition

$$(2)\qquad \forall \varepsilon > 0\ \exists O(a)\ \exists n_0 \in \mathbb{N}\ \ni\ \forall n > n_0\ \forall x \in O(a) \cap K :\ \varrho(f_n(x), f(x)) < \varepsilon.$$

For sequences $(f_n)_{n\geq 1}$ which converge pointwise on X to the function f, definition (1) is obviously equivalent with definition (2). Because of this we shall suppose in the sequel that the convergence domain K is the whole space X (when it is not specified otherwise).

The following theorem is a generalization of Lemma 5 of [3] p.149.

Theorem 1 *Let $f_n \to f$ on X and suppose that $(f_n)_{n\geq 1}$ converges uniformly at a point $a \in X$. If infinitely many functions f_n are continuous at a, then f is continuous at a.*

PROOF. Let $\varepsilon > 0$. The uniform convergence at the point a yields

$$\exists O(a)\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ \forall x \in O(a):\ \varrho(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

According to our assumption we can choose $s > n_0$ such that f_s is continuous at the point a. Therefore

$$\exists V(a)\ \forall x \in V(a):\ \varrho(f_s(x), f_s(a)) < \frac{\varepsilon}{3}.$$

Then for each $x \in O(a) \cap V(a)$ we have:

$$\varrho(f(a), f(x)) \leq \varrho(f(a), f_s(a)) + \varrho(f_s(a), f_s(x)) + \varrho(f_s(x), f(x)) < \\ < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The continuity of f at a follows. □

Let us compare the relationship between the locally uniform convergence and the uniform convergence at a point. The locally uniform convergence at a point a of a sequence $(f_n)_{n\geq 1}$ which converges on X to the function f means the validity of the assertion

$$\exists O(a)\ \forall \varepsilon > 0\ \exists n_0\ \forall n > n_0\ \forall x \in O(a):\ \varrho(f_n(x), f(x)) < \varepsilon. \tag{3}$$

When we compare this formula with (2) which is equivalent to Definition 2, we see that uniform convergence at a point is weaker than the locally uniform convergence at the point. Let us denote by $U(f_n)_{n\geq 1}$ or $U(F)$ the set of all points $a \in X$ at which the sequence $(f_n)_{n\geq 1} =: F$ converges uniformly. Comparing the two definitions we see that

$$LU(f_n)_{n\geq 1} \subset U(f_n)_{n\geq 1} \tag{4}$$

holds for each sequence $(f_n)_{n\geq 1}$. These two sets can be very different. The following example shows that inclusion in (4) can be strict even for continuous functions.

Example 1 *We shall construct a sequence of functions* $(f_n)_{n\geq 1}$ *which will uniformly converge at each irrational number* $x \in [0,1]$ *and, will not converge locally uniformly at any point.*

PROOF. Recall the well-known Riemann function $R : [0,1] \to \mathbb{R}$ defined as follows: $R(x) = \frac{1}{q}$ if x is rational number, $x = \frac{p}{q}$ (the irreducible form of x) and, $R(x) = 0$ if x is irrational number.

Let $(r_n)_{n\geq 1}$ be one-to-one sequence of all rational numbers of the interval $[0,1]$. For every $n \geq 1$ put

$$d_n := \frac{1}{2}\min\{|r_i - r_j| : 1 \leq i < j \leq n\}.$$

Now we can define the n-th function as follows

$$f_n(x) = \begin{cases} 0, & \text{if } x \in [0,1] \setminus \bigcup_{i=1}^{n} [r_i - d_n, r_i + d_n], \\ \left[1 - \frac{|x-r_i|}{d_n}\right] R(r_i), & \text{if } x \in [r_i - d_n, r_i + d_n]. \end{cases}$$

The sequence $(f_n)_{n\geq 1}$ converges pointwise to R and the behaviour of $(f_n)_{n\geq 1}$ near the points r_i $(i = 1, 2, \dots)$ is analogous to the behaviour of the sequence $(x^n)_{n\geq 1}$ nearby the point 1. Because of this, the sequence $(f_n)_{n\geq 1}$ does not converge locally uniformly at any rational number. Therefore this sequence does not converge locally uniformly at any point $x \in [0,1]$ as in each neighbourhood of x there is a rational number. We show that in each irrational number $x \in [0,1]$, the sequence converges uniformly. Let $\varepsilon > 0$ and $x \in [0,1]$ be an irrational number. Choose an $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\varepsilon}{2}$. Choose a $k_0 \in \mathbb{N}$ such that the finite sequence $r_1, r_2, \dots, r_{k_0}$ contains all $\frac{p}{q} \in [0,1]$ with $q \leq m$. Further we choose a δ such that $0 < \delta < \min\{|x - r_i|/2 : 1 \leq i \leq k_0\}$, so the distance of the sets $\{r_1, r_2, \dots, r_{k_0}\}, (x - \delta, x + \delta)$ is positive. Since $d_n \downarrow 0$, there exists a $k_1 > k_0$ such that $d_{k_1} < \delta$. Now let $k > k_1$ and $y \in (x - \delta, x + \delta)$. If there is an $i \leq k$ such that $y \in [r_i - d_k, r_i + d_k]$, then $i > k_0$ so $R(r_i) < \frac{1}{m}$. By simple estimation we get

$$|f_k(y) - R(y)| = |(1 - \frac{|y - r_i|}{d_k})R(r_i) - R(y)| \leq R(r_i) + R(y) \leq \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon.$$

In the case if y is not in any segment $[r_i - d_k, r_i + d_k]$ with $1 \leq i \leq k$ then

$$|f_k(y) - R(y)| = R(y) < \frac{1}{m} < \frac{\varepsilon}{2} < \varepsilon.$$

Thus the sequence $(f_n)_{n\geq 1}$ converges uniformly at x. □

The concept of quasiuniform convergence of sequences of functions is well-known (cf. [11], p.143). This concept plays an important part in formulation of conditions for the continuity of limit functions of sequences of continuous functions. By the analogy with Definition 2 we give the following local version of quasiuniform convergence.

Definition 3 *A sequence of functions* $(f_n)_{n\geq 1}$ *is said* **to converge quasi-uniformly at a point** a **(to the function** f **)** if $f_n(a) \to f(a)$ and the following assertion holds: $\forall \varepsilon > 0\ \forall m \geq 0\ \exists O(a)\ \exists p \geq 1\ \forall x \in O(a)$:

$$\min\{\varrho(f_{m+1}(x), f(x)), \dots, \varrho(f_{m+p}(x), f(x))\} < \varepsilon.$$

We show that the concept of quasiuniform convergence at a point enables us to give a very weak sufficient condition for continuity of limit function.

Theorem 2 *If all functions* f_n $(n \geq 1)$ *are continuous at a point* a *and the sequence* $(f_n)_{n\geq 1}$ *converges quasiuniformly at the point* a *to the function* f *then the function* f *is continuous at* a.

PROOF. Let $\varepsilon > 0$. Since $f(a) = \lim_{n\to\infty} f_n(a)$, there is an integer $m \in \mathbb{N}$ such that

$$\forall n > m: \ \varrho(f(a), f_n(a)) < \frac{\varepsilon}{3}. \tag{5}$$

Applying the quasiuniform convergence of $(f_n)_{n\geq 1}$ at the point a there is a neighbourhood $O(a)$ such that

$$\exists p \geq 1\ \forall x \in O(a)\ \exists k \leq p: \ \varrho(f_{m+k}(x), f(x)) < \frac{\varepsilon}{3}. \tag{6}$$

The continuity of finite number of functions $f_{m+1}, f_{m+2}, \dots, f_{m+p}$ at the point a implies that

$$\exists V(a)\ \forall x \in V(a) \cap K\ \forall i \leq p: \ \varrho(f_{m+i}(a), f_{m+i}(x)) < \frac{\varepsilon}{3}. \tag{7}$$

Let x be an arbitrary point from $O(a) \cap V(a)$. Owing to (6) we can choose $k \leq p$ such that

$$\varrho(f_{m+k}(x), f(x)) < \frac{\varepsilon}{3}. \tag{8}$$

Then using (7), (5) and the triangle inequality for ϱ we get

$$\begin{aligned}\varrho(f(a), f(x)) &\leq \varrho(f(a), f_{m+k}(a)) + \varrho(f_{m+k}(a), f_{m+k}(x)) + \varrho(f_{m+k}(x), f(x)) \\ &< 3\frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

The continuity of f at a follows. $\square$

3 Topological classification of sets of locally uniform and uniform convergence points

In the first section of this paper we have introduced the sets $LU(f_n)_{n\geq 1}$ and $U(f_n)_{n\geq 1}$. In connection with Definition 3 we put

$$Q(f_n)_{n\geq 1} := \{x \in X : (f_n)_{n\geq 1} \text{ converges quasiuniformly at } x\}.$$

In this section we shall investigate the topological structure of these sets.

The topological structure of the set $LU(f_n)_{n\geq 1}$ of all points of locally uniform convergence is simple as the next proposition shows.

Proposition 2 *For any sequence $(f_n : X \to Y)_{n\geq 1}$ the set $LU(f_n)_{n\geq 1}$ is open relative to the convergence domain $K \subset X$.*

PROOF. If $a \in LU(f_n)_{n\geq 1}$ then there is an open neighbourhood $O(a)$ such that $f_n \rightrightarrows f$ *on* $O(a) \cap K$. For each point $x \in O(a) \cap K$ the set $O(a) \cap K$ is a neighbourhood of the point x in the space K and therefore x belongs to $LU(f_n)_{n\geq 1}$ as well. Consequently we have

$$O(a) \cap K \subset LU(f_n)_{n\geq 1},$$

so a is an interior point of $LU(f_n)_{n\geq 1}$ relative to K. □

The construction of the sequence in Example 1 enables us to prove the following result which can be considered as a converse of Proposition 2 in a special case.

Proposition 3 *For each open set $G \subset \mathbb{R}$ there is a sequence $(f_n : \mathbb{R} \to \mathbb{R})_{n\geq 1}$ convergent on $\mathbb{R}$ such that $LU(f_n)_{n\geq 1} = G$.*

PROOF. We can write $G = \bigcup_{i=1}^{\infty} (a_i, b_i)$, with $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$. On the set $S = \mathbb{R} \setminus \bigcup_{i=1}^{\infty} [a_i, b_i]$ we can define the function g_n by the same method as f_n was constructed in Example 1. Over each interval (a_i, b_i) we can define the "trapezoid-like" function for each $n \geq 1$ as follows:

$$h_{ni}(x) := \begin{cases} \min\{2, A_1, A_2\} & \text{if } -\infty < a_i < x < b_i < +\infty \\ \min\{2, B\} & \text{if } -\infty = a_i < x < b_i < +\infty \\ \min\{2, C\} & \text{if } -\infty < a_i < x < b_i = +\infty \\ 2 & \text{if } -\infty = a_i < x < b_i = +\infty \end{cases}$$

where

$$A_1 = R(a_i) + \frac{2 - R(a_i)}{b_i - a_i} 2^n (x - a_i), R(b_i) + \frac{2 - R(b_i)}{b_i - a_i} 2^n (b_i - x) = A_2,$$
$$B = R(b_i) + (2 - R(b_i)) 2^n (b_i - x), \text{ and}$$
$$C = R(a_i) + (2 - R(a_i)) 2^n (x - a_i).$$

Now we can define the function

$$f_n(x) = \begin{cases} h_{ni}(x), & \text{if } x \in (a_i, b_i) \text{ for some } i \\ g_n(x), & \text{for } x \in S. \end{cases}$$

Then it is easy to verify that $LU(f_n)_{n\geq 1} = G$. □

It easily follows from Definitions 2 and 3 that for each sequence $(f_n)_{n\geq 1}$ which converges pointwise on X, the inclusion

$$U(f_n)_{n\geq 1} \subset Q(f_n)_{n\geq 1} \tag{9}$$

holds. The following example shows that these sets can be very different.

Example 2 *Let $X = Y = \mathbb{R}$ and let $(r_1, r_2, \ldots)$ be a one-to-one sequence of all rational numbers. Let us define the functions $f_n : \mathbb{R} \to \mathbb{R}$ $(n = 1, 2, \ldots)$ as follows*

$$f_n(x) = \begin{cases} 1, & \text{for } x = r_n \\ 0, & \text{for } x \neq r_n. \end{cases}$$

Obviously $f_n \longrightarrow f_0$ pointwise on $\mathbb{R}$, where $f_0(x) = 0$ for each $x \in \mathbb{R}$. Note that $U(f_n)_{n\geq 1} = \emptyset$. Indeed, this is a simple consequence of the fact that each neighbourhood of an arbitrary point $x \in \mathbb{R}$ contains the rational numbers r_m with arbitrarily large indices m and for $n \neq m$ we have

$$|f_m(r_m) - f_n(r_m)| = 1.$$

We prove that $Q(f_n)_{n\geq 1} = \mathbb{R}$. Observe that for each number $x_0 \in \mathbb{R}$, each its neighbourhood $V(x_0)$ and arbitrary number $m \in \mathbb{N}$, we have

$$\min\{|f_{m+1}(x) - f_0(x)|, |f_{m+2}(x) - f_0(x)|\} < \varepsilon$$

since the left-hand side of the inequality is zero in both cases: $x = r_{m+1}$ or $x \neq r_{m+1}$. Thus we get

$$\emptyset = U(f_n)_{n\geq 1} \subset Q(f_n)_{n\geq 1} = \mathbb{R}.$$

We now introduce the concept of the oscillation of a sequence of functions. This concept is a modification of a similar idea introduced by Hobson in his book [4] pp.133-135. We show that the structure of sets $U(f_n)_{n\geq 1}$ can be described by using the notion of oscillation of sequences of functions.

Let $F := (f_n)_{n\geq 1}$ be a sequence of functions on a topological space X with values in a metric space Y. For each point $a \in X$ and an open neighbourhood $V(a) =: V$ and every natural number N we define the number

$$\omega_F(a, V, N) := \sup_{\substack{m,n\geq N \\ x\in V}} \varrho(f_m(x), f_n(x)).$$

We omit the sign F of the sequence if no confusion is possible. If $N < N'$ then obviously $\omega(a, V, N') \leq \omega(a, V, N)$. Therefore there exists

$$\omega(a, V) := \lim_{N\to\infty} \omega(a, V, N) = \inf_{N\geq 1} \omega(a, V, N).$$

Definition 4 *Let $\mathcal{U}(a)$ be the system of all open neighbourhoods of point a. The number*

$$\omega_F(a) =: \omega(a) := \inf_{V\in\mathcal{U}(a)} \omega(a, V)$$

is said to be the **oscillation of the sequence $F = (f_n)_{n\geq 1}$ at the point a.**

Using the oscillation of a sequence we can give a characterization of the points of uniform convergence.

Theorem 3 *A sequence $F := (f_n)_{n\geq 1}$ convergent pointwise on X converges uniformly at a point $a \in X$ if and only if $\omega(a) = 0$.*

PROOF. If the sequence F converges uniformly at a point a then according to Definition 3 the next assertion is true:

$$\forall\, \varepsilon > 0\ \exists V \in \mathcal{U}(a)\ \exists N\ \forall m, n \geq N\ \forall x \in V: \quad \varrho(f_m(x), f_n(x) < \varepsilon.$$

From this we get immediately: $\omega(a, V, N) \leq \varepsilon$, $\omega(a, V) \leq \varepsilon$, $\omega(a) \leq \varepsilon$. As $\varepsilon > 0$ was arbitrarily small we have $\omega(a) = 0$.

Conversely if $\omega(a) = 0$. then for each $\varepsilon > 0$, there is a neighbourhood V of a such that $\omega(a, V) < \varepsilon$. By the definition of $\omega(a, V)$ there is a natural number N such that $\omega(a, V, N) < \varepsilon$. Thus we have established the validity of the assertion

$$\forall \varepsilon > 0\ \exists V\ \exists N \in \mathbb{N}\ \forall m, n > N\ \forall x \in V: \ \varrho(f_n(x), f_m(x)) < \varepsilon.$$

This implies the uniform convergence of $(f_n)_{n\geq 1}$ at the the point a. □

Corollary 1 *For each sequence $F = (f_n)_{n \geq 1}$ with the convergence domain X we have*

$$U(F) = \{x \in X : \omega_F(x) = 0\}.$$

Lemma 1 *Let $\eta > 0$, and $F = (f_n : X \to Y)_{n \geq 1}$ be a sequence of functions (not necessarily continuous). Then the set*

$$M_\eta := \{x \in X : \omega_F(x) < \eta\}$$

is open in X.

PROOF. We shall prove that each point $x_0 \in M_\eta$ is an interior point of M_η. Since $\omega(x_0) < \eta$, there is a neighbourhood V of x_0 such that $\omega(x_0, V) < \eta$. By the definition of $\omega(x_0, V)$ there is an integer $N \geq 1$ such that

$$\omega(x_0, V, N) = \sup_{\substack{m,n \geq N \\ x \in V}} \varrho(f_m(x), f_n(x)) < \eta.$$

Since for each $x \in V$ the open set V is neighbourhood of x, we have

$$\omega(x, V, N) < \eta, \ \omega(x, V) < \eta, \ \omega(x) < \eta$$

for each $x \in V$. Hence $V \subset M_\eta$ which shows that x_0 is an interior point of the set M_η and consequently M_η is open set. □

Lemma 2 *For each sequence $F := (f_n : X \to Y)_{n \geq 1}$ with the convergence domain X, the set $U(F)$ is of type G_δ in X.*

PROOF. As a consequence of Theorem 3 we have

$$U(F) = \bigcap_{k=1}^{\infty} U_k,$$

where $U_k := \{x \in X : \omega(x) < \frac{1}{k}\}$, $k \geq 1$. It follows from Lemma 14 that each U_k is an open set in X and the assertion follows. □

We have seen in Example 2 that the set $U(F)$ can be empty. Therefore, the problem appears when $U(F) \neq \emptyset$. We show that this is the case when X is a complete metric space.

Theorem 4 *Let (X, d) be complete metric space and (Y, ϱ) be a metric space. Suppose that all functions $f_n : X \to Y$, $(n \geq 1)$ are continuous on X and converge pointwise on X. Then $U(f_n)_{n \geq 1}$ is a dense set in X.*

PROOF.

1) In the first part of the proof we shall show that $U(f_n)_{n\geq 1}$ is non empty. For each positive integer N put

$$E_N := \{x \in X \mid \forall m, n \geq N : \varrho(f_m(x), f_n(x) \leq \varepsilon\}.$$

Since $(f_n)_{n\geq 1}$ converges pointwise on X, we have

$$X = \bigcup_{N=1}^{\infty} E_N.$$

Owing to the continuity of functions f_n, each set E_N $(N \geq 1)$ is closed. Applying the Baire category theorem to the complete space X, there exist a number N such that E_N is not nowhere dense in X, it means $intE_N \neq \emptyset$. Therefore there is an open ball $B(a,\delta)$, $(\delta > 0)$ such that

$$B(a,\delta) \subset E_N. \tag{10}$$

We can choose in the previous consideration $\varepsilon = 1$ and using (10) there is a number N_1 and a ball $B(a_1,\delta_1)$ in (X,d) such that

$$B(a_1,\delta_1) \subset E_{N_1},$$

so we have

$$\forall m, n \geq N_1 \ \forall x \in B(a_1,\delta_1) : \ \varrho(f_m(x), f_n(x)) \leq 1.$$

We can repeat the previous construction considering instead of X the complete subspace

$$X_1 = \overline{B(a_1, \frac{\delta_1}{2})} \subset X.$$

For this complete space there is a number N_2 and a ball (in X_1) $B(a_2,\delta_2)$ with $\delta_2 \leq \frac{\delta_1}{2}$, for which

$$B(a_2,\delta_2) \subset \overline{B(a_1, \frac{\delta_1}{2})}$$

and such that the assertion

$$\forall m, n \geq N_2 \ \forall x \in B(a_2,\delta_2) : \ \varrho(f_m(x), f_n(x)) \leq \frac{1}{2}$$

holds. In this way (by induction) we can construct a decreasing sequence of balls

$$B(a_1,\delta_1) \supset B(a_2,\delta_2) \supset \cdots \supset B(a_k,\delta_k) \supset B(a_{k+1},\delta_{k+1}) \supset \ldots,$$

such that

$$B(a_{k+1},\delta_{k+1}) \subset \overline{B(a_k,\frac{\delta_k}{2})}, \quad \delta_{k+1} \le \frac{\delta_k}{2} \ (\forall k \in \mathbb{N}),$$

and a sequence $(N_k)_{k\geq 1}$ of positive integers such that

$$\forall m,n \geq N_k \ \forall x \in B(a_k,\delta_k): \ \varrho(f_m(x),f_n(x)) \le \frac{1}{k}.$$

According to the well-known Cantor theorem there exists a point

$$a \in \bigcap_{j=1}^{\infty} \overline{B(a_j,\delta_j)}.$$

(cf. [11] p.82, Exercise 1). We show that $a \in U(f_n)_{n\geq 1}$. Let $\varepsilon > 0$. We take a natural number k such that $\frac{1}{k} < \varepsilon$. Since $a \in \overline{B(a_{k+1},\delta_{k+1})} \subset B(a_k,\delta_k)$, the ball $B(a_k,\delta_k)$ is a neighbourhood of the point a and the following assertion is true

$$\forall x \in B(a_k,\delta_k) \ \forall m,n \geq N_k: \ \varrho(f_m(x),f_n(x)) \le \frac{1}{k} < \varepsilon.$$

So the sequence $(f_n)_{n\geq 1}$ converges uniformly at the point a.

2) Now we prove the density of $U(f_n)_{n\geq 1}$ in each open ball $B(p,\delta)$ in X. The set $Z := \overline{B(p,\frac{\delta}{2})}$ is a closed metric subspace of X. Applying the first part of the proof to the complete metric space (Z,d) and to the functions $g_n := f_n|Z$, we can find a point $q \in Z$ such that $(g_n)_{n\geq 1}$ converges uniformly at the point q. From this the uniform convergence of $(f_n)_{n\geq 1}$ at q follows immediately. Hence $U(f_n)_{n\geq 1}$ has the nonempty intersection with each ball in X and therefore it is dense in X and the theorem follows. □

It is well-known that a dense G_δ set in complete metric space is residual (cf.[7] p.49). Combining Theorem 4 with Lemma 2 and the condition (9) we obtain the following corollary.

Corollary 2 *Under the assumptions of Theorem 4 each of the sets $U(f_n)_{n\geq 1}$ and $Q(f_n)_{n\geq 1}$ is dense and residual in X.*

4 Applications to points of continuity, strong differentiability and uniform differentiability

Combining Corollary 2 with Theorem 1 we obtain a well-known theorem on continuity points of functions in the Baire class one.

Corollary 3 *Let X be a (non-void) complete metric space and Y be a metric space. If the functions $f_n : X \to Y$ $(n \geq 1)$ are continuous, and converge pointwise to f then the set C_f (the set of all continuity points of f) is residual in X.*

It follows from Theorem 1 that under the assumptions of Corollary 3 the inclusion $U(f_n)_{n\geq 1} \subset C_f$ holds. This inclusion can be strict as the next example shows.

Example 3 *Let us choose $X = Y = \mathbb{R}$ with the Euclidean metric, and $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \in X$. Then the pointwise limit f of the sequence is zero function and we have $0 \in C_f$. But at the point 0 the sequence does not converge uniformly. This is because for $\varepsilon = \frac{1}{3}$ and for each neighbourhood $U(0)$ there are points of the form $x = \pm 1/n$ in which we have $|f_n(x) - f(x)| = |f_n(\pm\frac{1}{n})| = \frac{1}{2} \not< \varepsilon$. Therefore the convergence at the point 0 is not uniform.*

Let us recall the concept of strong differentiability of functions which has been studied by many authors. (cf. [2], [5]). A function $f : [a,b] \to \mathbb{R}$ is **strongly differentiable at a point** $x_0 \in [a,b]$ iff there exists the finite double limit

$$\lim_{\substack{x\to x_0\\ h\to 0}} \frac{f(x+h)-f(x)}{h} = f^*(x_0) \in \mathbb{R}.$$

The number $f^*(x_0)$ is called the strong derivative of f at x_0. The set of all points of strong differentiability of f will be denoted by $D^*(f)$. A function $f : [a,b] \to \mathbb{R}$ can be extended to the interval $[b, b+1]$ putting $f(x) = f(b)$ for $b \leq x \leq b+1$. Then for each $n = 1, 2, \ldots$ and every $x \in [a,b]$ we can define the functions $f_n(x) := n(f(x+\frac{1}{n}) - f(x))$. These functions can be used to characterize the set $D^*(f)$.

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and denote by D the set of all differentiability points of f. Then $D^*(f) = D \cap U(f_n)_{n\geq 1}$.*

PROOF. The inclusion "$\subset$" is true because for each point $x_0 \in [a,b]$ the equality

$$\lim_{\substack{x\to x_0\\ h\to 0}} \frac{f(x+h)-f(x)}{h} = f^*(x_0)$$

implies

$$\lim_{\substack{x\to x_0\\ n\to \infty}} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} = f^*(x_0)$$

and it means that the sequence $(f_n)_{n\geq 1}$ converges uniformly at the point x_0 to the derivative $f^*(x_0)$. To prove the inclusion "$\supset$" let us suppose that

$(f_n)_{n\geq 1}$ converges uniformly at a point $x_0 \in D$. By our assumption that f is continuous, each function $f_n (n \geq 1)$ is continuous at x_0 and so is the derivative f'. The continuity of f' at the point x_0 implies the strong differentiability of f at the point x_0 (see [2] Th.2). □

From Theorem 5 and the Lemma 2 we get the following result.

Corollary 4 *If $f : [a,b] \to \mathbb{R}$ is a continuous function then the set $D^*(f)$ of all points of strong differentiability of the function is a G_δ subset of the set D of all differentiability points of f.*

Corollary 5 *Let $f : [a,b] \to \mathbb{R}$ be differentiable function. Then the set $D^*(f)$ of all strong differentiability points of f is residual in $[a,b]$.*

PROOF. The differentiability of f implies that the sequence $f_n := n(f(x + \frac{1}{n}) - f(x))$ converges on $[a,b]$ and it follows from Corollary 2 that $U(f_n)_{n\geq 1}$ is residual in $[a,b]$. Since $D^*(f) = U(f_n)_{n\geq 1}$ the theorem follows. □

When f is a function differentiable on an interval $[a,b]$ then, according to [10], the strong differentiability is equivalent to the **uniform differentiability** introduced in [8] and therefore the set of all points of uniform differentiability of f is equal to $D^*(f)$. So we get the following result of [8],Th. 4.

Corollary 6 *If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ then the set of all points of uniform differentiability of f is residual set in $[a,b]$.*

The notion of symmetric differentiability is well-known. In the paper [5] the authors introduce the notion of uniform symmetric differentiability of a function f. Let us denote by $UD^s(f)$ the set of all points of uniform symmetric differentiability of function $f : [a,b] \to \mathbb{R}$. We can give a new proof of the following theorem which was proved by S.N.Mukhopadhyay in [9] (cf. also [5] Th.II)

Theorem 6 *If $f : [a,b] \to \mathbb{R}$ is a continuous symmetrically differentiable function then the set $UD^s(f)$ of all points of uniform symmetric differentiability of f is residual set in $[a,b]$.*

PROOF. Let us continuously extend the function f to the interval $[a-1, b+1]$ defining $f(x) = f(a)$ if $a - 1 \leq x \leq a$ and $f(x) = f(b)$ if $b \leq x \leq b+1$. If we define the sequence

$$f_n(x) := \frac{f(x+\frac{1}{n}) - f(x - \frac{1}{n})}{\frac{2}{n}} \quad (n \geq 1)$$

then $(f_n)_{n\geq 1}$ converges pointwise on $[a,b]$ to the symmetric derivative f^s. Due Corollary 2 the set $U(f_n)_{n\geq 1}$ is residual in $[a,b]$. Owing to Theorem 1

f^s is continuous at every point of the set $U(f_n)_{n\geq 1}$. But f is uniformly symmetrically differentiable at each point of continuity of f^s (see [5] Th. 1.2). So we have $U(f_n)_{n\geq 1} \subset UD^s(f)$ and theorem follows owing to the residuality of the set $U(f_n)_{n\geq 1}$. □

It is proved in [10] that a continuous symmetrically differentiable function has the strong derivative at a point if an only if it has the uniform symmetric derivative at the point. Thus we have a stronger version of Corollary 5.

Corollary 7 *If $f : [a,b] \to \mathbb{R}$ is continuous symmetrically differentiable function then the set of all points of strong differentiability of f is residual set in $[a,b]$.*

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