Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 736–740

Josip E. Pečarić, Department of Mathematics, University of Zagreb, 41000 Zagreb, Croatia
Gh. Toader, Department of Mathematics, Technical University, 3400 Cluj-Napoca, Romania

THE PRESERVATION OF THE CONVEXITY OF FUNCTIONS

1 Introduction

Let us consider the classes of continuous, convex, starshaped and superadditive functions defined respectively by:

$$\begin{aligned} C(b) &= \{f : [0,b] \to \mathbb{R},\ f(0) = 0,\ f \text{ continuous}\} \\ K(b) &= \{f \in C(b) \mid f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \\ &\qquad \forall t \in (0,1),\ \forall x, y \in [0,b]\} \\ St(b) &= \{f \in C(b) \mid f(tx) \le tf(x),\ \forall t \in [0,1],\ x \in [0,b]\} \\ S(b) &= \{f \in C(b) \mid f(x+y) \ge f(x) + f(y),\ \forall x, y, x+y \in [0,b]\}. \end{aligned}$$

In [2] it is proved that all these classes are preserved by the arithmetic integral mean A defined by

$$A(f)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \text{ for } x > 0,\ A(f)(0) = 0.$$

Moreover, if for a given set F of functions we set

$$MF = \{f \in C(b) \mid A(f) \in F\},$$

in [2] it is proved that for any positive b the following strict inclusions hold:

$$K(b) \subset MK(b) \subset St(b) \subset S(b) \subset MSt(b) \subset MS(b).$$

Simple proofs of these relations are also given in [5].

Mathematical Reviews subject classification: 26A51 and 26D15
Received by the editors February 14, 1994

References [3] and [4] consider the integral operator W_g, defined by

$$W_g(f)(x) = \frac{1}{g(t) - g(0)} \int_0^x g'(x) f(t)\, dt, \quad W_g(f)(0) = f(0) \tag{1}$$

where g is a given differentiable function. In [5] it is proved that if W_g preserves one of the classes $K(b)$, $St(b)$, or $S(b)$, then the function g is necessarily of the form $g(x) = kx^u$ for some $u > 0$ and some $k \neq 0$. If we denote the resulting operator by A_u

$$A_u(f)(x) = \frac{u}{x^u} \int_0^x t^{u-1} f(t)\, dt \tag{2}$$

and if for a given set F of functions we set $M^u F = \{f \in C(b) : A_u(f) \in F\}$, then it is proved that for any positive numbers b and u the following inclusions hold:

$$\begin{array}{ccccccc} K(b) & \subset & M^u K(b) & \subset & St(b) & \subset & S(b) \\ & & & & \cap & & \cap \\ & & & & M^u St(b) & \subset & M^u S(b) \end{array}$$

A similar result was proved for some classes of generalized convexity of order two in [6] and [7] and for convexity, starshapedness and superadditivity of higher order in [8].

Analyzing all these results, we can produce a general scheme that we want to consider in what follows.

2 A Class of Generalized Convex Functions

Let $D = (d_{jk})_{n,m}$ be a $n \times m$ matrix and $C = (c_j)_n$ be a given n vector with the property that $c_1 + \cdots + c_n = 0$. Let

$$D(b) = \left\{ X = (x_k)_m \mid \sum_{k=1}^{m} d_{jk} x_k \in [0, b],\ j = 1, \ldots, n \right\}$$

and then, for any X from $D(b)$, the functional $L_{CD}(\cdot)(X) : C(b) \to \mathbb{R}$ defined by

$$L_{CD}(f)(X) = \sum_{j=1}^{n} c_j f \left\{ \sum_{k=1}^{m} d_{jk} x_k \right\}$$

Using them, we can define a general class of convex functions

$$K_{CD}(b) = \{ f \in C[0, b] \mid L_{CD}(f)(X) \geq 0,\ \forall X \in D(b) \}.$$

By adequate choices of C and D we get the sets of Jensen convex functions and of superadditive functions, usual or generalized, and of any order. For example the condition of superadditivity of $f \in C[0,b]$ is

$$f(x_1 + x_2) - f(x_1) - f(x_2) + f(0) \geq 0, \;\; \forall\, x_1, x_2, x_1 + x_2 \in [0,b]$$

and it becomes that given in the definition of $S(b)$ for f from $C(b)$. In [8] we considered also superadditivity of order $n > 2$. For example $f \in C[0,b]$ is said to be superadditive of order 3 if

$$\begin{aligned} f(x_1 + x_2 + x_3) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3) + f(x_1) + \\ + f(x_2) + f(x_3) - f(0) \geq 0, \;\; \forall\, x_1, x_2, x_3, x_1 + x_2 + x_3 \in [0,b]. \end{aligned}$$

For convexity and starshapedness we must refer to Remark 3.

The condition on C assures that the class $K_{CD}(b)$ is nonempty because it contains the constant functions. But we need a more precise condition. For this let us denote by P_q the set of polynomials of degree at most q.

Definition 1 *The class $K_{CD}(b)$ is well defined if there is an integer $q \geq 1$ such that $L_{CD}(f) = 0$ if and only if $f \in P_q$.*

Remark 1 The determination of the value of q for C and D given is a problem of functional equations. Of course, necessary conditions are $L_{CD}(e_k) = 0$ for $k = 0, \ldots, q$, and $L_{CD}(e_{q+1}) \neq 0$ where $e_k(x) = x^k$ for $k \geq 0$. But it is a difficult problem to prove that they are also sufficient or to find simpler conditions. For some results and references see [1, pages. 129–131]. For example if $L_{CD}(f)(X) = \sum_{j=1}^{n} c_j f(x_1 + (j-1)x_2)$, the value of q is less than 1 plus the order of multiplicity of the root $t = 1$ in the equation $c_1 + c_2 t + \cdots + c_n t^{n-1} = 0$.

3 Main Results

We want to determine those functions g that give an integral operator W_g, defined by (1), which preserves the class $K_{CD}(b)$. We have the following result.

Theorem 1 *If the class of functions $K_{CD}(b)$ is well defined and W_g preserves it, then there is a positive number u such that $g(x) = vx^u \; \forall\, x \in [0,b]$.*

PROOF. For any p from P_q, because p and $-p$ belong to $K_{CD}(b)$, we have $W_g(p)$ and $W_g(-p)$ also in $K_{CD}(b)$ and this is equivalent to $L_{CD}(W_g(p))(X) = 0 \; \forall\, X \in D(b)$. Thus $W_g(p)$ is in P_q as $K_{CD}(b)$ is well defined. Let $W_g(e_k) = p_k$ for $k = 1, \ldots, q$. Differentiating these relations we get

$$\frac{g'(x)}{g(x) - g(0)} = \frac{p'_k}{e_k(x) - p_k(x)} \text{ for } x \in (0,b], \; k = 1, \ldots, q \tag{3}$$

or, if we set $p_k(x) = a_{k0} + a_{k1}x + \cdots + a_{kq}x^q$, we have for $1 \leq k < h \leq q$

$$\left(x^h - \sum_{j=0}^{q} a_{hj}x^j\right) \sum_{j=1}^{q} ja_{kj}x^{j-1} = \left(x^k - \sum_{j=1}^{q} a_{kj}x^j\right) \sum_{j=1}^{q} ja_{hj}x^{j-1}.$$

For $h = q$ equating the coefficients of x^{2q-1} we get $a_{kq} = 0$ for $k < q$. Then for $h = q-1$ and the power $2q-3$, we deduce also $a_{k,q-1} = 0$ for $k < q-1$ and by induction $a_{kj} = 0$ for $k < j$. Thus $p_1(x) = a_{10} + a_{11}x$ and from (3), with $k = 1$, we have $\frac{g'(x)}{g(x) - g(0)} = \frac{a_{11}}{x - (a_{10} + a_{11}x)}$ which gives the result. □

Using such a weight function we denote the resulting operator by A_u. It is given by (2). Also we introduce the following class of functions

$$M^u K_{CD}(b) = \{f \in C(b) \mid A_u(f) \in K_{CD}(b)\}.$$

Theorem 2 *If tX belongs to $D(b)$ for any $t \in [0,1]$ and any $X \in D(b)$, then for any positive u we have $K_{CD}(b) \subset M^u K_{CD} b$.*

PROOF. Substituting $t = xs^{1/u}$ in $A_u(f)$ we get $A_u(f)(x) = \int_0^1 f(xs^{1/u})ds$. So for any X from $D(b)$

$$\begin{aligned} L_{CD}(A_u(f))(X) &= \sum_{j=1}^{n} c_j A_u(f)\left(\sum_{k=1}^{m} d_{jk}x_k\right) \\ &= \sum_{j=1}^{n} c_j \int_0^1 f\left(s^{1/u} \sum_{k=1}^{m} d_{jk}x_k\right) ds \\ &= \int_0^1 \sum_{j=1}^{n} c_j f\left(\sum_{k=1}^{m} d_{jk}x_k s^{1/u}\right) ds \\ &= \int_0^1 L_{CD}(f)(Xs^{1/u})ds \geq 0 \end{aligned}$$

because f is from $K_{CD}(b)$ and $s^{1/u}X$ from $D(b)$.

Remark 2 *The condition $[0,1] \times D(b) \subset D(b)$ holds, for example, if the matrix D is positive.*

Remark 3 *If instead C and D we use families of vectors C and of matrices D, all the above results remain valid. So we obtain similar theorems for various sets of convex or of starshaped functions. For example, the function $f \in C[0,b]$ is starshaped if $tf(x) - f(tx) + (1-t)f(0) \geq 0 \ \forall x \in [0,b]$ for every $t \in [0,1]$, that is, we have a set of conditions.*

References

[1] J. Aczél, *Lectures on functional equations and their applications*, New York, London, 1966.

[2] A. M. Bruckner and E. Ostrow, *Some function classes related to the class of convex functions*, Pacific J. Math. **12** (1962), 1203–1215.

[3] I. Lacković, *On convexity of arithmetic integral mean*, Univ. Beograd. Publ. Elektrotehn. Fak. 381–409 (1972), 117–120.

[4] C. Mocanu, *Monotony of weight-means of higher order*, Anal. Numér. Théor. Approx. **1** (1982), 115–127.

[5] Gh. Toader, *On the hierarchy of convexity of functions*, Anal. Numér. Théor. Approx. **15** (1986), 167–172.

[6] Gh. Toader, *On a general type of convexity*, Studia Univ. Babes-Bolyai, Math. **31** (1986), **4**, 37–40.

[7] Gh. Toader, *On a generalization of the convexity*, Mathematica **30** (53), (1988), 83–87.

[8] Gh. Toader, *A hierarchy of convexity of higher order of functions*, Mathematica **35** (58), (1993), **1**, 93–98.