Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 673–680

Aleksander Maliszewski*, Department of Mathematics, Pedagogical University, Chodkiewicza 30, 85-064 Bydgoszcz, Poland, e-mail: wspb05@pltumk11.bitnet

SUMS OF BOUNDED DARBOUX FUNCTIONS

Abstract

In this article I prove a few theorems concerning representations of functions in a given family of functions with a common bound as the sums of bounded Darboux functions with a common summand, which are analogous to the results of Fast, Mišík and Pu & Pu.

Let $\mathfrak{A}$ be a family of real functions. Consider the following condition:

$a(\mathfrak{A})$: there exists a function f such that $f+g$ is Darboux for each $g \in \mathfrak{A}$.

It was remarked by A. Lindenbaum [10] that $a(\{h, 0\})$ holds for each real function h. This result was generalized by H. Fast [9] in the following manner: $a(\mathfrak{A})$ holds for each family $\mathfrak{A}$ with $\operatorname{card}\mathfrak{A} \le \mathfrak{c}$. Moreover, $a(\mathfrak{A})$ holds for some families $\mathfrak{A}$ with $\operatorname{card}\mathfrak{A} = 2^{\mathfrak{c}}$, e.g., for the family of all Lebesgue measurable functions and the family of all functions with the Baire property [13]. (Cf. also [4].) Evidently, $a(\mathfrak{A})$ does not hold if $\mathfrak{A}$ is the family of all real functions.

An analogous result was proved by L. Mišík [12]. (Cf. also [4].)

Theorem 1 *If $\mathfrak{A}$ is a countable family of Baire α functions and $\alpha > 1$, then there exists a Baire α function f such that $f+g$ has the Darboux property for every $g \in \mathfrak{A}$.*

It was shown by H. W. Pu and H. H. Pu in 1987 that the above theorem does not hold in case $\alpha = 1$. However, it is still true if $\mathfrak{A}$ is finite [14].

Theorem 2 *Let $\mathfrak{A}$ be a finite family of Baire one functions. Then there exists a Baire one function f such that $f+g$ is Darboux for every $g \in \mathfrak{A}$.*

Key Words: Darboux function, Baire class α, Baire property, $\mathfrak{c}$-dense set
Mathematical Reviews subject classification: Primary 26A15. Secondary 26A21, 54C08
Received by the editors November 11, 1994
*Supported by a KBN Research Grant 2 1144 91 01, 1992–94.

In this article I study analogous problems but assume moreover that the functions in $\mathfrak{A}$ have a common bound and require the function f to be bounded.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$, the set of rationals by $\mathbb{Q}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. The symbol χ_A means the characteristic function of a set $A \subset \mathbb{R}$. The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. For any function f we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbb{R}\}$, for each non-degenerate interval I we define $\mathfrak{c}\text{-sup}(f, I) = \sup\{t \in \mathbb{R} : \operatorname{card}[f^{-1}((t, \infty)) \cap I] = \mathfrak{c}\}$ and we let $\mathfrak{c}\text{-inf}(f, I) = -\,\mathfrak{c}\text{-sup}(-f, I)$. Clearly $-\|f\| \leq \mathfrak{c}\text{-inf}(f, I) \leq \mathfrak{c}\text{-sup}(f, I) \leq \|f\|$.

The *oscillation of a function f on a non-empty set $A \subset \mathbb{R}$* will be denoted by $\omega(f, A)$ (i.e., $\omega(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$). Similarly, the *oscillation of a function f at a point $x \in \mathbb{R}$* will be denoted by $\omega(f, x)$ (i.e., $\omega(f, x) = \lim_{r \to 0^+} \omega(f, (x - r, x + r))$).

Let κ be a cardinal number. We shall use the following set theoretical assumptions.

$E_1(\kappa)$: The union of κ sets of cardinality less than $\mathfrak{c}$ has cardinality less than $\mathfrak{c}$.
$E_2(\kappa)$: The union of κ null subsets of $\mathbb{R}$ is a null set.
$E_3(\kappa)$: The union of κ first category subsets of $\mathbb{R}$ is of the first category.

It is well-known that for each $\kappa < \mathfrak{c}$ these assumptions follow from the Martin's Axiom and therefore also from the Continuum Hypothesis (see, e.g., [15]).

In the proofs of the main results we will need a few lemmas.

Lemma 3 *Let C be a first category subset of an open interval (a, b) and let λ be a positive extended real number. There exists a Darboux Baire one function h defined on (a, b) which vanishes except on a first category null set disjoint from C and such that* $\liminf_{x \to a^+} h(x) = \liminf_{x \to b^-} h(x) = -\lambda$ *and* $\limsup_{x \to a^+} h(x) = \limsup_{x \to b^-} h(x) = \lambda$. [2, Lemma 2]

Theorem 4 *Let f be a Baire one function. Then the following two conditions are equivalent:*

1. *f is Darboux;*
2. *for each $x \in \mathbb{R}$ there exist sequences $y_n \nearrow x$ and $z_n \searrow x$ such that*

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(z_n) = f(x).$$

[1, Theorem 1.1, p. 9]

Lemma 5 *Suppose $\kappa \leq \mathfrak{c}$ and let $\{K_\alpha : \alpha < \kappa\}$ be a family of subsets of $\mathbb{R}$ of cardinality $\mathfrak{c}$. Then we can find a family $\{Q_\alpha : \alpha < \kappa\}$ of pairwise disjoint sets of cardinality $\mathfrak{c}$ such that $Q_\alpha \subset K_\alpha$ for each $\alpha < \kappa$. Moreover, if each of the* *sets K_α is Lebesgue measurable (resp. has the Baire property), then we can require that each of the sets Q_α has measure zero (resp. is nowhere dense).*

PROOF. We use transfinite induction. For each $\beta < \mathfrak{c}$ and each $\alpha < \min\{\beta, \kappa\}$ choose an $x_{\beta,\alpha} \in K_\alpha \setminus \left(\{x_{\gamma,\delta} : \gamma < \beta,\ \delta < \min\{\gamma, \kappa\}\} \cup \{x_{\beta,\delta} : \delta < \alpha\}\right)$.

For each $\alpha < \kappa$ define $Q_\alpha = \{x_{\beta,\alpha} : \alpha < \beta < \mathfrak{c}\}$. It is easy to show that the family $\{Q_\alpha : \alpha < \kappa\}$ possesses desired properties.

Finally observe that each Lebesgue measurable set (resp. each set with the Baire property) of cardinality $\mathfrak{c}$ contains a subset of measure zero (resp. a nowhere dense subset) of cardinality $\mathfrak{c}$, so we may assume that each of the sets K_α is a null set (resp. a nowhere dense set). Then we proceed as above.□

Now we can prove the main results of the article.

Theorem 6 *Let $\kappa < \mathfrak{c}$. Assume $E_1(\kappa)$ and let $\{g_\alpha : \alpha < \kappa\}$ be a family of real functions with $\sup\{\|g_\alpha\| : \alpha < \kappa\} = M < \infty$. Then we can find a function f such that $\|f\| \le 2M$ and $f + g_\alpha$ is Darboux for each $\alpha < \kappa$. If we moreover assume $E_2(\kappa)$ (resp. $E_3(\kappa)$) and all functions g_α are Lebesgue measurable (resp. have the Baire property), then we can require that $f \neq 0$ only on a null set (resp. that $f \neq 0$ only on a set of the first category).*

PROOF. Let $\{I_n : n \in \mathbb{N}\}$ be an enumeration of all open intervals with rational end points. Fix an $n \in \mathbb{N}$ and an $\alpha < \kappa$. There is a non-decreasing sequence $\{a_{n,\alpha,l} : l \in \mathbb{Z}\}$ with limit points $\mathfrak{c}\text{-inf}(g_\alpha, I_n)$ and $\mathfrak{c}\text{-sup}(g_\alpha, I_n)$ such that

$$(\bullet) \qquad \operatorname{card}\left(g_\alpha^{-1}([a_{n,\alpha,l}, a_{n,\alpha,l+1}]) \cap I_n\right) = \mathfrak{c}$$

for each $l \in \mathbb{Z}$.

Indeed, let $\beta = \mathfrak{c}\text{-sup}(g_\alpha, I_n)$. Note that for each $\varepsilon > 0$,

$$g_\alpha^{-1}([\beta - \varepsilon, \infty)) \cap I_n = \bigcup_{n=1}^{\infty} \left(g_\alpha^{-1}([\beta - \frac{\varepsilon}{n}, \beta - \frac{\varepsilon}{n+1}]) \cap I_n\right) \cup \left(g_\alpha^{-1}(\beta) \cap I_n\right)$$
$$\cup \bigcup_{n=1}^{\infty} \left(g_\alpha^{-1}([\beta + \frac{\varepsilon}{n+1}, \beta + \frac{\varepsilon}{n}]) \cap I_n\right) \cup \left(g_\alpha^{-1}([\beta + \varepsilon, \infty)) \cap I_n\right).$$

By the definition of $\mathfrak{c}\text{-sup}(g_\alpha, I_n)$, the sets in the lower line have cardinality strictly less than $\mathfrak{c}$. So either $\operatorname{card}\left(g_\alpha^{-1}([\beta - 1/n, \beta - 1/(n+1)])\right) = \mathfrak{c}$ for infinitely many n, or $\operatorname{card}\left(g_\alpha^{-1}(\beta)\right) = \mathfrak{c}$. In the first case $\mathfrak{c}\text{-inf}(g_\alpha, I_n) < \beta$, and we can easily find $\mathfrak{c}\text{-inf}(g_\alpha, I_n) < a_{n,\alpha,0} < a_{n,\alpha,1} < \cdots < \beta$ fulfilling $(\bullet)$; otherwise we define $a_{n,\alpha,0} = a_{n,\alpha,1} = \cdots = \beta$. Similarly we can construct $a_{n,\alpha,0} \ge a_{n,\alpha,-1} \ge \cdots \ge \mathfrak{c}\text{-inf}(g_\alpha, I_n)$. By Lemma 5, there is a family $\{Q_{n,\alpha,l} : n \in \mathbb{N}, \alpha < \kappa, l \in \mathbb{Z}\}$ of pairwise disjoint sets of cardinality $\mathfrak{c}$ such that for each $n \in \mathbb{N}$, $\alpha < \kappa$ and $l \in \mathbb{Z}$

$$Q_{n,\alpha,l} \subset g_\alpha^{-1}\left([a_{n,\alpha,l}, a_{n,\alpha,l+1}]\right) \cap I_n.$$

Moreover we can require that each of the sets $Q_{n,\alpha,l}$ is a null set if all functions g_α are Lebesgue measurable, and that each of the sets $Q_{n,\alpha,l}$ is nowhere dense if all functions g_α have the Baire property.

Fix an $\alpha < \kappa$. Denote by A_α the set of all $x \in \mathbb{R}$ such that

$$g_\alpha(x) \notin \big[\mathfrak{c}\text{-inf}(g_\alpha, I), \mathfrak{c}\text{-sup}(g_\alpha, I)\big]$$

for some non-degenerate interval $I \ni x$. In a standard way one can easily prove that $\operatorname{card} A_\alpha < \mathfrak{c}$. (See also [7, Lemma 4] and [5, Lemma 1].) Hence by $E_1(\kappa)$, $\operatorname{card} A < \mathfrak{c}$, where $A = \bigcup_{\alpha<\kappa} A_\alpha$.

For each $n \in \mathbb{N}$, $\alpha < \kappa$ and $l \in \mathbb{Z}$ find a non-empty bilaterally $\mathfrak{c}$-dense in itself set $P_{n,\alpha,l} \subset Q_{n,\alpha,l} \setminus A$ and construct a function $h_{n,\alpha,l}$ which vanishes outside of $P_{n,\alpha,l}$ and which maps $P_{n,\alpha,l}$ onto

$$\big(-2M + a_{n,\alpha,l+1}, 2M + a_{n,\alpha,l}\big).$$

Define the function f by formula

$$f(x) = \begin{cases} h_{n,\alpha,l}(x) - g_\alpha(x) & \text{if } x \in P_{n,\alpha,l},\ n \in \mathbb{N},\ \alpha < \kappa,\ l \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Then a straightforward calculation gives $|f| < 2M$ on $\mathbb{R}$.

Fix an $\alpha < \kappa$ and let I be a non-degenerate interval. We will show that $(f + g_\alpha)(I)$ is an interval with end points $a = -2M + \mathfrak{c}\text{-inf}(g_\alpha, I)$ and $b = 2M + \mathfrak{c}\text{-sup}(g_\alpha, I)$.

Let $x \in I$. If $x \in A_\alpha$, then $f(x) = 0$ and $f(x) + g_\alpha(x) \in [-M, M] \subset [a, b]$.

Otherwise $g_\alpha(x) \in \big[\mathfrak{c}\text{-inf}(g_\alpha, I), \mathfrak{c}\text{-sup}(g_\alpha, I)\big]$, so $f(x) + g_\alpha(x) \in (a, b)$. It follows that $(f + g_\alpha)(I) \subset [a, b]$.

On the other hand, since for any function g

$$\mathfrak{c}\text{-sup}(g, I) = \sup\big\{\mathfrak{c}\text{-sup}(g, I_n) : I_n \subset I,\ n \in \mathbb{N}\big\},$$

so

$$(a, b) = \bigcup_{I_n \subset I} \bigcup_{l \in \mathbb{Z}} h_{n,\alpha,l}(P_{n,\alpha,l}) \subset (f + g_\alpha)(I).$$

From the above we infer that $f + g_\alpha$ is a Darboux function. To complete the proof note that $\big\{x \in \mathbb{R} : f(x) \neq 0\big\} \subset \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha<\kappa} \bigcup_{l\in\mathbb{Z}} P_{n,\alpha,l}$. □

Theorem 7 *Assume that $\mathfrak{A}$ is a countable family of Baire α functions such that* $\sup\{\|g\| : g \in \mathfrak{A}\} = M < \infty$. *Then we can find a Baire 2 function f such that $\|f\| \leq 2M$, $f + g$ is Darboux for each $g \in \mathfrak{A}$ and $f \neq 0$ only on a first category null set.*

PROOF. The proof is a repetition of the argument used in Theorem 6. The difference is that: (1) we use Lemma 2 of [16] instead of Lemma 5 to find non-empty perfect sets $Q_{n,\alpha,l}$, (2) for all $n, \alpha \in \mathbb{N}$ and $l \in \mathbb{Z}$ we find a non-empty perfect subset $P_{n,\alpha,l}$ of the set of points of bilateral condensation of $Q_{n,\alpha,l} \setminus A$ (observe that since the family $\mathfrak{A}$ is countable, the set A is countable, too; moreover, we can assume that $P_{n,\alpha,l}$ is a null set and by [11], that $g_\alpha | P_{n,\alpha,l}$ is continuous), (3) we require that each function $h_{n,\alpha,l}$ is in the first class of Baire [3]. □

Theorem 8 *Assume that $\mathfrak{A}$ is a finite family of Baire one functions such that $\sup\{\|g\| : g \in \mathfrak{A}\} = M < \infty$. Then we can find a Baire one function f such that $\|f\| \leq 2M$, $f + g$ is Darboux for each $g \in \mathfrak{A}$ and $f \neq 0$ only on a first category null set.*

PROOF. Denote by D the union of the sets of points of discontinuity of all functions from $\mathfrak{A}$. Set $\lambda_n = M \cdot 2^{1-n}$ for $n \in \mathbb{N} \cup \{0\}$.

Fix an $n \in \mathbb{N}$. Let $A_n = \{x \in \mathbb{R} : \omega(g, x) \geq \lambda_n \text{ for some } g \in \mathfrak{A}\}$. (Note that A_n is closed and nowhere dense.) Find an isolated set $B_n \subset \mathbb{R} \setminus D$ such that $A_n \cup B_n$ is closed and each $x \in A_n$ is a point of bilateral accumulation of B_n. Use Lemma 3 for each component of the complement of $A_n \cup B_n$ to construct a Darboux Baire one function $h_n : \mathbb{R} \to [-\lambda_{n-1}, \lambda_{n-1}]$ such that $h_n = 0$ except on a first category set of measure zero which is disjoint from $D \cup \bigcup_{i<n}\{x \in \mathbb{R} : h_i(x) \neq 0\}$ and such that for each $a \in A_n \cup B_n$

$$\liminf_{x \to a^+} h_n(x) = \liminf_{x \to a^-} h_n(x) = -\lambda_{n-1}$$

and

$$\limsup_{x \to a^+} h_n(x) = \limsup_{x \to a^-} h_n(x) = \lambda_{n-1}.$$

Define $f = \sum_{n=1}^{\infty} h_n$. Since this series is uniformly convergent, f is a Baire one function. Clearly $\|f\| \leq 2M$ and $f \neq 0$ only on a first category null set, so we have to prove that $f + g$ is Darboux for each $g \in \mathfrak{A}$ to complete the proof.

Take a $g \in \mathfrak{A}$. The function $f + g$ is in the first class of Baire, so we can use Young's condition (i.e., condition 2. of Theorem 4).

Fix an $x \in \mathbb{R}$. If $x \in D$, then $f(x) = 0$ and $x \in A_m \setminus A_{m-1}$ for some $m \in \mathbb{N}$ (we let $A_0 = \emptyset$). Choose a sequence $\widetilde{y}_n \nearrow x$ of elements of B_m such that the limit $\widetilde{\lambda} = \lim_{n\to\infty} g(\widetilde{y}_n)$ does exist. Since $x \notin A_{m-1}$, so $|g(x) - \widetilde{\lambda}| \leq \lambda_{m-1}$. Let $t_n \longrightarrow 0$ be such that $|g(x) - g(\widetilde{y}_n) + t_n| < \lambda_{m-1}$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we can find a $y_n \in (\widetilde{y}_n, \widetilde{y}_{n+1})$ such that $|g(y_n) - g(\widetilde{y}_n)| < 1/n$ and $f(y_n) = g(x) - g(\widetilde{y}_n) + t_n$. Then evidently $y_n \nearrow x$ and

$$\lim_{n\to\infty} \left(f(y_n) + g(y_n)\right) = g(x) = f(x) + g(x).$$

Now let $x \notin D$. It is easy to show that there exists a sequence $y_n \nearrow x$ such that $\lim_{n\to\infty} f(y_n) = f(x)$. Then

$$\lim_{n\to\infty} \left(f(y_n) + g(y_n)\right) = \lim_{n\to\infty} f(y_n) + \lim_{n\to\infty} g(y_n) = f(x) + g(x).$$

Similarly we proceed to show that there exists a sequence $z_n \searrow x$ such that $\lim_{n\to\infty}\left(f(z_n) + g(z_n)\right) = f(x) + g(x)$. This completes the proof. □

The below corollary is an affirmative answer to a question by J. Ceder [6].

Corollary 9 *Suppose $\mathfrak{B}$ is one of the following families of functions: all functions, all Lebesgue measurable functions, all functions with the Baire property, Baire class α. Then for each bounded function $g \in \mathfrak{B}$ we can find bounded Darboux functions $f_1, f_2 \in \mathfrak{B}$ such that $g = f_1 + f_2$.*

PROOF. It suffices to use one of the above theorems for the two-element family of functions $\mathfrak{A} = \{g, 0\}$. □

Remark. It is quite evident that there is an unbounded function g such that $f + g$ is Darboux for no bounded function f. (Consider, e.g., the function $\sum_{n=1}^{\infty} n\chi_{\{n\}}$.) The following example shows that we cannot drop the assumption $E_1(\kappa)$ in Theorem 6.

Example 1 Let κ be a cardinal number. If $E_1(\kappa)$ fails, then there is a family of functions $\mathfrak{A}$ such that: (1) $\operatorname{card}\mathfrak{A} = \kappa$, (2) $\|g\| \le 1$ for each $g \in \mathfrak{A}$, (3) for each bounded above function f there is a $g \in \mathfrak{A}$ such that $f+g$ is not Darboux.

Indeed, let $\{A_\alpha : \alpha < \kappa\}$ be a family of sets of cardinality less than $\mathfrak{c}$ such that $\operatorname{card}\bigcup_{\alpha<\kappa} A_\alpha = \mathfrak{c}$. Clearly we may assume that these sets are pairwise disjoint and $\bigcup_{\alpha<\kappa} A_\alpha = \mathbb{R}$. Let $\mathfrak{A} = \{\chi_{A_\alpha} : \alpha < \kappa\}$.

Let f be bounded above. Let $x \in \mathbb{R}$ be such that $f(x) > M$, where $M = \sup\{f(t) : t \in \mathbb{R}\} - 1$. There exists an $\alpha < \kappa$ with $x \in A_\alpha$. Then $f(t) + \chi_{A_\alpha}(t) = f(t) \le M + 1$ for $t \notin A_\alpha$ and $f(x) + \chi_{A_\alpha}(x) > M + 1$, so

$$1 \le \operatorname{card}\{t \in \mathbb{R} : f(t) + \chi_{A_\alpha}(t) > M + 1\} \le \operatorname{card} A_\alpha < \mathfrak{c}$$

and $f + \chi_{A_\alpha}$ is not Darboux. □

The next example shows that we cannot allow infinite family $\mathfrak{A}$ in Theorem 8. (The functions in the example in [14] have no common bound.)

Example 2 For any function f, if $f + \chi_{\{q\}}$ has the Darboux property for each $q \in \mathbb{Q}$, then f is discontinuous everywhere (so it is not a Baire one function).

Indeed, let f be continuous at some $x \in \mathbb{R}$. Choose an $\eta > 0$ such that $|f(t) - f(x)| < 1/2$ for $t \in (x - \eta, x + \eta)$ and let $q \in \mathbb{Q} \cap (x - \eta, x + \eta)$.

Then $f(q) + \chi_{\{q\}}(q) > f(x) + 1/2$ and since $f(t) + \chi_{\{q\}}(t) < f(x) + 1/2$ for $t \in (x - \eta, x + \eta) \setminus \{q\}$, so $f + \chi_{\{q\}}$ is not Darboux. $\square$

Finally I would like to present a query. It is well known that each function which is almost continuous in the sense of Stallings possesses the Darboux property, and that the converse need not be true. (Cf., e.g., [13].) However, each function can be written as the sum of two almost continuous functions. It would be of interest to know whether the summand functions can be chosen bounded provided that the given function is bounded. (During the first Joint US-Polish Workshop in Real Analysis, Łódź 1994, P. Humke and U. Darji showed that each bounded function can be written as the sum of <u>three</u> bounded almost continuous functions [8]. The question whether <u>two</u> summands will do is still open.) The other theorems of this article yield analogous questions.

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lect. Notes in Math., vol. 659, Springer Verlag, 1978.

[2] A. M. Bruckner, J. G. Ceder, and R. Keston, *Representations and approximations by Darboux functions in the first class of Baire*, Rev. Roumaine Math. Pures Appl. **13** (1968), no. 9, 1247–1254.

[3] A. M. Bruckner and J. L. Leonard, *Stationary sets and determining sets for certain classes of Darboux functions*, Proc. Amer. Math. Soc. **16** (1965), 935–940.

[4] J. G. Ceder, *On representing functions by Darboux functions*, Acta Sci. Math. (Szeged) **26** (1965), 283–288.

[5] ______, *Differentiable roads for real functions*, Fund. Math. **65** (1969), 351–358.

[6] ______, *A necessary and sufficient condition for a Baire α function to be a product of two Darboux Baire α functions*, Rend. Circ. Mat. Palermo (2) **36** (1987), 78–84.

[7] J. G. Ceder and T. L. Pearson, *Insertion of open functions*, Duke Math. J. **35** (1968), 277–288.

[8] U. B. Darji and P. D. Humke, *Every bounded function is the sum of three almost continuous bounded functions*, to appear in Real Anal. Exchange.

[9] H. Fast, *Une remarque sur la propriété de Weierstrass*, Colloq. Math. **7** (1959), 75–77.

[10] A. Lindenbaum, *Sur quelques propriétés des fonctions de variable réelle*, Ann. Soc. Math. Polon. **6** (1927), 129–130.

[11] I. Maximoff, *Sur les fonctions ayant la propriété de Darboux*, Prace Mat.-Fiz. **43** (1936), 241–265.

[12] L. Mišík, *Zu zwei Sätzen von W. Sierpiński*, Rev. Roumaine Math. Pures Appl. **12** (1967), 849–860.

[13] T. Natkaniec, *Almost continuity*, Real Anal. Exchange **17** (1991–92), no. 2, 462–520.

[14] H. W. Pu and H. H. Pu, *On representations of Baire functions in a given family as sums of Baire Darboux functions with a common summand*, Časopis Pěst. Mat. **112** (1987), no. 3, 320–326.

[15] J. Shoenfield, *Martin's axiom*, Amer. Math. Monthly **82** (1975), no. 6, 610–617.

[16] J. Smítal, *On approximation of Baire functions by Darboux functions*, Czechoslovak Math. J. **21** (1971), no. 96, 418–423.