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CARDINAL INVARIANTS CONCERNING FUNCTIONS WHOSE SUM IS ALMOST CONTINUOUS

Abstract

Let $\mathcal{A}$ stand for the class of all almost continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let $A(\mathcal{A})$ be the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no $g\colon \mathbb{R} \to \mathbb{R}$ with the property that $f + g \in \mathcal{A}$ for all $f \in F$. We define cardinal number $A(\mathcal{D})$ for the class $\mathcal{D}$ of all real functions with the Darboux property similarly. It is known, that $\mathfrak{c} < A(\mathcal{A}) \leq 2^{\mathfrak{c}}$ [11]. We will generalize this result by showing that the cofinality of $A(\mathcal{A})$ is greater that $\mathfrak{c}$. Moreover, we will show that it is pretty much all that can be said about $A(\mathcal{A})$ in ZFC, by showing that $A(\mathcal{A})$ can be equal to any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$ and that it can be equal to $2^{\mathfrak{c}}$ independently of the cofinality of $2^{\mathfrak{c}}$. This solves a problem of T. Natkaniec [11, Problem 6.1, p. 495].

We will also show that $A(\mathcal{D}) = A(\mathcal{A})$ and give a combinatorial characterization of this number. This solves another problem of Natkaniec. (Private communication.)

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1 Preliminaries.

We will use the following terminology and notation. Functions will be identified with their graphs. The family of all functions from a set X into Y will be denoted by Y^X. Symbol $|X|$ will stand for the cardinality of a set X. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathfrak{c}$. For a cardinal number κ we will write $\mathrm{cf}(\kappa)$ for the cofinality of κ. A cardinal number κ is regular, if $\kappa = \mathrm{cf}(\kappa)$. Recall also, that the Continuum Hypothesis (abbreviated as CH) stands for the statement $\mathfrak{c} = \aleph_1$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *almost continuous* (in the sense of Stallings [14]) if and only if for every open set $U \subseteq \mathbb{R}^2$ containing f there exists a continuous function $g \subseteq U$. So, every neighborhood of f in the graph topology contains a continuous function. This concept was introduced by Stallings [14] in connection with fixed points. We will use symbol $\mathcal{A}$ to denote the family of almost continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

For $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ define the cardinal $\mathrm{A}(\mathcal{F})$ as follows:

$$\begin{aligned} \mathrm{A}(\mathcal{F}) &= \min\{|F|\colon F \subseteq \mathbb{R}^{\mathbb{R}} \,\&\, \neg\exists g \in \mathbb{R}^{\mathbb{R}}\, \forall f \in F\ f+g \in \mathcal{F}\} \\ &= \min\{|F|\colon F \subseteq \mathbb{R}^{\mathbb{R}} \,\&\, \forall g \in \mathbb{R}^{\mathbb{R}}\, \exists f \in F\ f+g \notin \mathcal{F}\} \end{aligned}$$

For a generalization of the next theorem see Natkaniec [11]. Fast [3] proved the same result for the family of Darboux functions.

Theorem 1.1 $\mathfrak{c} < \mathrm{A}(\mathcal{A}) \leq 2^{\mathfrak{c}}$.

□

At the Joint US–Polish Workshop in Real Analysis in Łódź, Poland, in July 1994 A. Maliszewski gave a talk mentioning several problems of his, Z. Grande and T. Natkaniec [4]. Natkaniec asked whether or not anything more could be said about the cardinal $\mathrm{A}(\mathcal{A})$. (See also Natkaniec [11, Problem 6.1, p. 495] or [12, Problem 1.7.1, p. 55].) In what follows we will show that pretty much nothing more can be said (in ZFC), except that the $\mathrm{cf}(\mathrm{A}(\mathcal{A})) > \mathfrak{c}$.

We will also study the family $\mathcal{D} \subseteq \mathbb{R}^{\mathbb{R}}$ of Darboux functions. Recall that a function is *Darboux* if and only if it takes every connected set to a connected set, or (in the case of a real function) satisfies the intermediate value property. Note that $\mathcal{A} \subseteq \mathcal{D}$. This is because if for example $f(a) < c < f(b)$ and c is omitted by f on (a,b), then take the h-shape set H (see Figure 1). The complement of H is an open neighborhood of the graph of f which does not contain a graph of a continuous function. It is known (Stallings [14]) that the inclusion $\mathcal{A} \subseteq \mathcal{D}$ is proper.

It is obvious from the definition that if $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R}^{\mathbb{R}}$ then $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$. In particular, $\mathrm{A}(\mathcal{A}) \leq \mathrm{A}(\mathcal{D})$. At the Joint US–Polish Workshop in Real Analysis

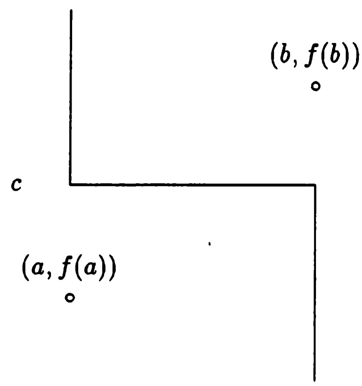

Figure 1: h-shape set H

in Łódź, Poland, in July 1994, T. Natkaniec asked the authors whether it is possible that $A(\mathcal{A}) < A(\mathcal{D})$. We will give a negative answer for this question by showing (in ZFC) that $A(\mathcal{A}) = A(\mathcal{D})$.

We will finish this section with the following technical fact, see Natkaniec [11, Thm. 1.2, p. 464].

Theorem 1.2 *(Kellum) There exists a family $\mathcal{B}$ of closed sets (called* a blocking family*) with the properties that:*

- *for every $f \in \mathbb{R}^{\mathbb{R}}$ we have*

$$f \in \mathcal{A} \text{ if and only if } \forall B \in \mathcal{B}\ f \cap B \neq \emptyset;$$

- *for every $B \in \mathcal{B}$ the projection $\mathrm{pr}_x(B)$ of B onto the x-axis (equivalently, the domain of B) is a non-degenerate interval.*

□

The paper is organized as follows. We will show that $A(\mathcal{D}) = A(\mathcal{A})$, give some other characterizations of this cardinal, and prove that $\mathrm{cf}(A(\mathcal{A})) > \mathfrak{c}$ in Section 2. In Section 3 we will prove that some forcing axioms imply that $A(\mathcal{A})$ can be any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$ and that $A(\mathcal{A})$ can be equal to $2^{\mathfrak{c}}$ for any value of $2^{\mathfrak{c}}$. The proof of the consistency of the forcing axioms used in Section 3 will be left for the Section 4.

2 $\mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{A})$ and its cofinality.

We will need the following definitions.

For a cardinal number $\kappa \leq \mathfrak{c}$ we define the family

$$\mathcal{D}(\kappa) \subseteq \mathbb{R}^{\mathbb{R}}$$

of κ *strongly Darboux functions* as the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathbb{R}$ the set $(a,b) \cap f^{-1}(y)$ has cardinality at least κ.

It is obvious from the definition that

$$\mathcal{D}(\lambda) \subseteq \mathcal{D}(\kappa) \text{ for all cardinals } \kappa \leq \lambda \leq \mathfrak{c}. \tag{1}$$

We will need the following lemmas.

Lemma 2.1 $\mathrm{A}(\mathcal{D}(\mathfrak{c})) > \mathfrak{c}$.

PROOF. Pick a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality continuum. We will find a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $f + g \in \mathcal{D}(\mathfrak{c})$ for all $f \in F$. Let

$$\langle \langle a_\xi, b_\xi, y_\xi, f_\xi \rangle : \xi < \mathfrak{c} \rangle$$

be an enumeration of the set of all

$$\langle a, b, y, f \rangle \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times F \text{ with } a < b,$$

such that each four-tuple appears in the sequence continuum many times. Define by induction a sequence $\langle x_\xi \in \mathbb{R} : \ \xi < \mathfrak{c} \rangle$ such that

$$x_\xi \in (a_\xi, b_\xi) \setminus \{x_\zeta : \zeta < \xi\}.$$

Then, any function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g(x_\xi) = y_\xi - f(x_\xi)$ for all $\xi < \mathfrak{c}$ has the property that $f + g \in \mathcal{D}(\mathfrak{c})$ for all $f \in F$. □

Lemma 2.2 $\mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{D}(\omega_1))$.

PROOF. Since $\mathcal{D}(\omega_1) \subseteq \mathcal{D}$ we have $\mathrm{A}(\mathcal{D}(\omega_1)) \leq \mathrm{A}(\mathcal{D})$. To prove the other inequality let $\kappa = \mathrm{A}(\mathcal{D}(\omega_1))$. Then, by (1) and Lemma 2.1,

$$\kappa = \mathrm{A}(\mathcal{D}(\omega_1)) \geq \mathrm{A}(\mathcal{D}(\mathfrak{c})) > \mathfrak{c}.$$

We will show that $\kappa \geq \mathrm{A}(\mathcal{D})$.

Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be a family of cardinality κ witnessing $\kappa = \mathrm{A}(\mathcal{D}(\omega_1))$:

$$\forall g \in \mathbb{R}^{\mathbb{R}}\ \exists f \in F\ \ f + g \notin \mathcal{D}(\omega_1). \tag{2}$$

It is enough to find family $F^* \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality κ such that

$$\forall g \in \mathbb{R}^{\mathbb{R}}\, \exists f^* \in F^*\; f^* + g \notin \mathcal{D}. \tag{3}$$

Define $F^* = \{h \in \mathbb{R}^{\mathbb{R}}\colon\ \exists f \in F\; h =^* f\}$, where $h =^* f$ if and only if the set $\{x\colon h(x) \neq f(x)\}$ is at most countable. Since $\kappa > \mathfrak{c}$ and for every $f \in \mathbb{R}^{\mathbb{R}}$ the set $\{h \in \mathbb{R}^{\mathbb{R}}\colon h =^* f\}$ has cardinality $\mathfrak{c}$, we have $|F^*| = \kappa$. It is enough to show that F^* satisfies (3). So, choose $g \in \mathbb{R}^{\mathbb{R}}$. Then, by (2), there exists $f \in F$ such that $f + g \notin \mathcal{D}(\omega_1)$. This means, that there are $a < b$ and $y \in \mathbb{R}$ such that the set $(a,b) \cap (f+g)^{-1}(y)$ is at most countable. Then we can find $f^* =^* f$ such that

- $(f^* + g)(a) < y$,
- $(f^* + g)(b) > y$, and
- $(f^* + g)(x) \neq y$ for every $x \in (a,b)$.

Thus, $f^* + g \notin \mathcal{D}$. □

Now, we are ready for one of our main theorems.

Theorem 2.1 $\mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{A})$.

PROOF. We already know that $\mathrm{A}(\mathcal{A}) \leq \mathrm{A}(\mathcal{D})$. So, by Lemma 2.2, it is enough to prove that $\mathrm{A}(\mathcal{D}(\omega_1)) \leq \mathrm{A}(\mathcal{A})$.

So, let $\kappa = \mathrm{A}(\mathcal{A})$. Then, by Theorem 1.1, $\kappa > \mathfrak{c}$ and, by the definition of $\mathrm{A}(\mathcal{A})$, there exists a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality κ witnessing it, i.e., such that

$$\forall g \in \mathbb{R}^{\mathbb{R}}\, \exists f \in F\; f + g \notin \mathcal{A}.$$

In particular, by the definition of the family $\mathcal{B}$ of blocking sets (from Theorem 1.2),

$$\forall g \in \mathbb{R}^{\mathbb{R}}\, \exists f \in F\; \exists B \in \mathcal{B}\; (f+g) \cap B = \emptyset. \tag{4}$$

It is enough to find a family $F^* \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality κ such that

$$\forall g \in \mathbb{R}^{\mathbb{R}}\, \exists f^* \in F^*\; f^* + g \notin \mathcal{D}(\omega_1). \tag{5}$$

In order to do this, choose a function $h_B \in \mathbb{R}^{\mathbb{R}}$ for every $B \in \mathcal{B}$ such that

$$(x, h_B(x)) \in B \quad \text{for every} \quad x \in \mathrm{pr}_x(B).$$

Let

$$F^* = \{f - h_B \colon f \in F \;\&\; B \in \mathcal{B}\}.$$

Clearly F^* has cardinality κ, since $|\mathcal{B}| \leq \mathfrak{c} < \kappa$. We will show that F^* satisfies (5). Let $g \in \mathbb{R}^{\mathbb{R}}$. Then, by (4), there exist $f \in F$ and $B \in \mathcal{B}$ such that $(f+g) \cap B = \emptyset$. In particular,

$$[(f - h_B) + g] \cap (B - h_B) = [(f+g) \cap B] - h_B = \emptyset,$$

where we define $Z - h_B = \{(x, y - h_B(x)) \colon (x,y) \in Z\}$ for any $Z \subseteq \mathbb{R}^2$. But $(B - h_B) \supset \mathrm{pr}_x(B) \times \{0\}$. Hence, $[(f - h_B) + g] \cap [\mathrm{pr}_x(B) \times \{0\}] = \emptyset$. In particular, $[(f - h_B) + g]^{-1}(0) \cap \mathrm{pr}_x(B) = \emptyset$. So, $f^* = f - h_B \in F^*$, while $(f - h_B) + g \notin \mathcal{D}(\omega_1)$ since, by Theorem 1.2, $\mathrm{pr}_x(B)$ contains a non-degenerate interval. □

To prove the next theorem we need a few more definitions. For a set $X \subseteq \mathbb{R}$ and a cardinal number $\kappa \leq \mathfrak{c}$ we define the family

$$\mathcal{D}(X, \kappa) \subseteq \mathbb{R}^X$$

as the family of all functions $f \colon X \to \mathbb{R}$ such that for all $a, b \in X$, $a < b$, and $y \in \mathbb{R}$ the set $(a,b) \cap f^{-1}(y)$ has cardinality at least κ. Similarly, define the cardinal $\mathrm{A}(\mathcal{F})$ as before:

$$\mathrm{A}(\mathcal{F}) \quad = \quad \min\{|F| \colon F \subseteq \mathbb{R}^X \,\&\, \forall g \in \mathbb{R}^X\ \exists f \in F\ f + g \notin \mathcal{F}\}$$

(Thus $\mathcal{D}(\mathbb{R}, \kappa) = \mathcal{D}(\kappa)$.) It is obvious from the definitions that for κ with $\omega_1 \leq \kappa \leq \mathfrak{c}$

$$\mathrm{A}(\mathcal{D}(\mathbb{R} \setminus \mathbb{Q}, \kappa)) = \mathrm{A}(\mathcal{D}(\mathbb{R}, \kappa)) \tag{6}$$

and also

$$\mathrm{A}(\mathcal{D}(X, \kappa)) = \mathrm{A}(\mathcal{D}(Y, \kappa)) \ \text{ for all order isomorphic } X, Y \subseteq \mathbb{R}. \tag{7}$$

Theorem 2.2 $\mathrm{A}(\mathcal{A}) = \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{D}(\mathfrak{c}))$.

PROOF. By (1) it is obvious that $\mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{D}(\omega_1)) \geq \mathrm{A}(\mathcal{D}(\mathfrak{c}))$.

To prove the other inequality let $F \in \mathbb{R}^{\mathbb{R}}$ be a family of cardinality κ with $\kappa < \mathrm{A}(\mathcal{D})$. It is enough to find $g \in \mathbb{R}^{\mathbb{R}}$ such that

$$f + g \in \mathcal{D}(\mathfrak{c}) \ \text{ for every } \ f \in F. \tag{8}$$

So, let $\langle S_\alpha \colon \alpha < \mathfrak{c} \rangle$ be a sequence of pairwise disjoint dense subsets of $\mathbb{R}$ each of which is order isomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers. By (6) and (7) for every $\alpha < \mathfrak{c}$ we have

$$\kappa < \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{D}(\omega_1)) = \mathrm{A}(\mathcal{D}(\mathbb{R}, \omega_1)) = \mathrm{A}(\mathcal{D}(\mathbb{R} \setminus \mathbb{Q}, \omega_1)) = \mathrm{A}(\mathcal{D}(S_\alpha, \omega_1)).$$

We can apply the definition of $\mathrm{A}(\mathcal{D}(S_\alpha,\omega_1))$ to the family

$$F|_{S_\alpha} = \{f|_{S_\alpha} \in \mathbb{R}^{S_\alpha} : f \in F\}$$

to find a function $g_\alpha \colon S_\alpha \to \mathbb{R}$ such that

$$(f|_{S_\alpha}) + g_\alpha \in \mathcal{D}(S_\alpha,\omega_1) \text{ for every } f \in F.$$

It is easy to see that any $g \in \mathbb{R}^\mathbb{R}$ extending $\bigcup_{\alpha<\mathfrak{c}} g_\alpha$ satisfies (8).

□

We will finish this section with one more cardinal equal to $\mathrm{A}(\mathcal{A})$. For any infinite cardinal κ let

$$\mathfrak{e}_\kappa = \min\{|F| \colon F \subseteq \kappa^\kappa \,\&\, \forall g \in \kappa^\kappa\, \exists f \in F\, |f \cap g| < \kappa\}.$$

This cardinal was extensively studied in Landver [7].

Theorem 2.3 $\mathrm{A}(\mathcal{A}) = \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathcal{D}(\mathfrak{c})) = \mathfrak{e}_\mathfrak{c}$.

PROOF. It is enough to prove that $\mathrm{A}(\mathcal{D}(\mathfrak{c})) = \mathfrak{e}_\mathfrak{c}$. It is also clear that

$$\mathfrak{e}_\mathfrak{c} = \min\{|F| \colon F \subseteq \mathbb{R}^\mathbb{R} \,\&\, \forall g \in \mathbb{R}^\mathbb{R}\, \exists f \in F\, |f \cap g| < \mathfrak{c}\}.$$

To prove the inequality $\mathrm{A}(\mathcal{D}(\mathfrak{c})) \le \mathfrak{e}_\mathfrak{c}$ let $F \subseteq \mathbb{R}^\mathbb{R}$ have cardinality $\kappa < \mathrm{A}(\mathcal{D}(\mathfrak{c}))$. Then, there exists $g \colon \mathbb{R} \to \mathbb{R}$ such that $g - f \in \mathcal{D}(\mathfrak{c})$ for every $f \in F$. In particular, $|(g-f)^{-1}(0)| = \mathfrak{c}$, i.e., $f(x) = g(x)$ for continuum many $x \in \mathbb{R}$. So, $|f \cap g| = \mathfrak{c}$ for all $f \in F$, i.e., $\kappa < \mathfrak{e}_\mathfrak{c}$. This proves $\mathrm{A}(\mathcal{D}(\mathfrak{c})) \le \mathfrak{e}_\mathfrak{c}$.

To prove $\mathfrak{e}_\mathfrak{c} \le \mathrm{A}(\mathcal{D}(\mathfrak{c}))$ take a family $F \subseteq \mathbb{R}^\mathbb{R}$ of cardinality $\kappa < \mathfrak{e}_\mathfrak{c}$. We will show that $\kappa < \mathrm{A}(\mathcal{D}(\mathfrak{c}))$.

Choose a sequence $\langle S^y_{a,b} \subseteq (a,b) \colon a,b,y \in \mathbb{R},\ a<b\rangle$ of pairwise disjoint sets of cardinality continuum. Applying $\kappa < \mathfrak{e}_\mathfrak{c}$ to the family

$$F^y_{a,b} = \{(y-f)|_{S^y_{a,b}} \colon f \in F\}$$

we can find $g^y_{a,b} \colon S^y_{a,b} \to \mathbb{R}$ such that $|(y-f)|_{S^y_{a,b}} \cap g^y_{a,b}| = \mathfrak{c}$ for every $f \in F$. In particular, $(y-f)(x) = g^y_{a,b}(x)$, i.e., $(f + g^y_{a,b})(x) = y$ for continuum many $x \in S^y_{a,b} \subseteq (a,b)$. Now, if we take any $g \in \mathbb{R}^\mathbb{R}$ extending $\bigcup\{g^y_{a,b} \colon a,b,y \in \mathbb{R},\ a<b\}$ then $(f+g)^{-1}(y) \cap (a,b)$ has cardinality continuum for every $f \in F$ and $a,b,y \in \mathbb{R}$, $a<b$. So, $\kappa < \mathrm{A}(\mathcal{D}(\mathfrak{c}))$. □

Corollary 2.1 $\mathrm{cf}(\mathrm{A}(\mathcal{A})) > \mathfrak{c}$.

PROOF. It is obvious that $\mathrm{cf}(\mathfrak{e}_\kappa) > \kappa$ since κ can be split into κ many sets of size κ. □

3 Forcing axioms and the value of $\mathrm{A}(\mathcal{A})$.

In this section we will prove the following two theorems.

Theorem 3.1 *Let $\lambda \geq \kappa \geq \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with ZFC that the Continuum Hypothesis ($\mathfrak{c} = \aleph_1$) is true, $2^{\mathfrak{c}} = \lambda$, and $\mathrm{A}(\mathcal{A}) = \kappa$.*

So for example if $2 \leq n \leq 17$, then it is consistent that

$$\mathfrak{c} = \aleph_1 < \mathrm{A}(\mathcal{A}) = \aleph_n \leq \aleph_{17} = 2^{\mathfrak{c}}.$$

Theorem 3.2 *Let λ be a cardinal such that $\mathrm{cf}(\lambda) > \omega_1$. Then it is relatively consistent with ZFC that the Continuum Hypothesis ($\mathfrak{c} = \aleph_1$) holds and $\mathrm{A}(\mathcal{A}) = \lambda = 2^{\mathfrak{c}}$.*

It follows from Theorem 3.2 that $\mathrm{A}(\mathcal{A})$ can be a singular cardinal, e.g. $\mathrm{A}(\mathcal{A}) = \aleph_{\omega_2}$ where $\mathfrak{c}^+ = \omega_2$. We do not know how to get $\mathrm{A}(\mathcal{A})$ strictly smaller than $2^{\mathfrak{c}}$ and singular.

The technique of proof is a variation on the idea of a Generalized Martin's Axiom (GMA). In this section we will formulate the forcing axioms and show that they imply the results. The proof of the consistency of these axioms will be left for Section 4.

For a partially ordered set $(\mathbb{P}, \leq)$ we say that $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-filter if and only if

- for all $p, q \in G$ there exists $r \in G$ with $r \leq p$ and $r \leq q$, and
- for all $p, q \in \mathbb{P}$ if $p \in G$ and $q \geq p$, then $q \in G$.

Define $D \subseteq \mathbb{P}$ to be dense if and only if for every $p \in \mathbb{P}$ there exists $q \in D$ with $q \leq p$.

For any cardinal κ and poset $\mathbb{P}$ define $\mathrm{MA}_\kappa(\mathbb{P})$ (Martin's Axiom for $\mathbb{P}$) to be the statement that for any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G such that $D \cap G \neq \emptyset$ for every $D \in \mathcal{D}$.

¿From now on, let $\mathbb{P}$ be the following partial order

$$\mathbb{P} = \{p \mid p : X \to \mathbb{R}, X \subseteq \mathbb{R}, \text{ and } |X| < \mathfrak{c}\}$$

i.e., the partial function from $\mathbb{R}$ to $\mathbb{R}$ of cardinality less than $\mathfrak{c}$. Define $p \leq q$ if and only if $q \subseteq p$, i.e., p extends q as a partial function.

Lemma 3.1 *$MA_\kappa(\mathbb{P})$ implies $\mathrm{A}(\mathcal{A}) \geq \kappa$.*

PROOF. We know by Theorem 2.3 that $A(\mathcal{A}) = \mathfrak{e}_{\mathfrak{c}} > \mathfrak{c}$. Thus, it is enough to prove that $MA_\kappa(\mathbb{P})$ implies $\mathfrak{e}_{\mathfrak{c}} \geq \kappa$ for $\kappa > \mathfrak{c}$. Note that for any $\mathbb{P}$-filter G since any two conditions in G must have a common extension, $\bigcup G$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$. Moreover, it is easy to see that for any $x \in \mathbb{R}$ the set

$$D_x = \{p \in \mathbb{P} : x \in \mathrm{dom}(p)\}$$

is dense in $\mathbb{P}$ and that $\bigcup G \colon \mathbb{R} \to \mathbb{R}$ for any $\mathbb{P}$-filter G intersecting all sets D_x.

Let $\langle S_\alpha : \alpha < \mathfrak{c}\rangle$ be a partition of $\mathbb{R}$ into pairwise disjoint sets of size $\mathfrak{c}$. Also for any $f \in \mathbb{R}^{\mathbb{R}}$ and $\alpha < \mathfrak{c}$ the set

$$D_{f,\alpha} = \{p \in \mathbb{P} : \exists x \in (\mathrm{dom}(p) \cap S_\alpha)\ \ p(x) = f(x)\}$$

is dense in $\mathbb{P}$. Given any $F \subseteq \mathbb{R}^{\mathbb{R}}$ with $|F| < \kappa$ let

$$\mathcal{D} = \{D_x : x \in \mathbb{R}\} \cup \{D_{f,\alpha} : f \in F, \alpha < \mathfrak{c}\}.$$

Notice that $|\mathcal{D}| = \mathfrak{c} < \kappa$. Applying $MA_\kappa(\mathbb{P})$ we can find a $\mathbb{P}$-filter G such that G meets every $D \in \mathcal{D}$. Letting $g = \bigcup G \colon \mathbb{R} \to \mathbb{R}$ we see that $|f \cap g| = \mathfrak{c}$ for every $f \in \mathcal{F}$. □

The proof of Lemma 3.1 is a kind of forcing extension of the inductive argument used in the proof of Theorem 1.1.

Notice also, that Theorem 3.2 follows immediately from Lemma 3.1, Theorem 1.1 and the following theorem.

Theorem 3.3 *Let λ be a cardinal such that $\mathrm{cf}(\lambda) > \omega_1$. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$ and that $MA_\lambda(\mathbb{P})$ holds.*

Thus, we have proved Theorem 3.2 modulo Theorem 3.3. Theorem 3.3 will be proved in Section 4.

Lemma 3.1 shows also one inequality of Theorem 3.1. To prove the reverse inequality we will use a different partial order $(\mathbb{P}^*, \leq)$. It is similar to $\mathbb{P}$ but in addition has some side conditions.

$$\mathbb{P}^* = \{(p, E) : p \in \mathbb{P} \text{ and } E \subseteq \mathbb{R}^{\mathbb{R}} \text{ with } |E| < \mathfrak{c}\}.$$

Define the ordering on $\mathbb{P}^*$ by

$$\begin{aligned}(p, E) \leq (q, F) \quad &\text{iff} \quad p \leq q \ \text{ and } \ E \supseteq F \\ &\text{and} \quad \forall x \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)\ \ \forall f \in F\ p(x) \neq f(x).\end{aligned}$$

The idea of the last condition is that we wish to create a generic function $g \in \mathbb{R}^{\mathbb{R}}$ with the property that for many f we have $g(x) \neq f(x)$ for almost all x. Thus, the condition (q, F) 'promises' that for all new x and old $f \in F$ it should be that $g(x) \neq f(x)$.

For a cardinal number κ define $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ to be the statement:

There exists a sequence $\langle G_\alpha : \alpha < \kappa \rangle$ of $\mathbb{P}^*$-filters, called a κ-Lusin sequence, such that for every dense set $D \subseteq \mathbb{P}^*$

$$|\{\alpha < \kappa : \ G_\alpha \cap D = \emptyset\}| < \kappa.$$

Thus we have a Lusin sequence of $\mathbb{P}^*$-filters. This is also known as a kind of Anti-Martin's Axiom. See vanDouwen and Fleissner [2], Miller and Prikry [10], Todorcevic [15], and Miller [9] for a similar axiom.

Lemma 3.2 *Suppose $\mathfrak{c} < \kappa$, κ is regular, and $Lus_\kappa(\mathbb{P}^*)$. Then* $\mathrm{A}(\mathcal{A}) \leq \kappa$.

PROOF. Let $\langle G_\alpha : \alpha < \kappa \rangle$ be a κ-Lusin sequence of $\mathbb{P}^*$-filters and let

$$g_\alpha = \bigcup \{p : \exists F \ (p, F) \in G_\alpha\}.$$

Then g_α is a partial function from $\mathbb{R}$ into $\mathbb{R}$. Similarly to the last proof, let

$$D_x = \{(p, F) \in \mathbb{P}^* : \ x \in \mathrm{dom}(p)\}.$$

To see that D_x is dense let (q, F) be an arbitrary element of $\mathbb{P}^*$ and suppose it is not already an element of D_x. The set $Q = \{f(x) : f \in F\}$ has cardinality less than $\mathfrak{c}$ so there exists $y \in \mathbb{R} \setminus Q$. Let $p = q \cup \{(x, y)\}$. Then $(p, F) \leq (q, F)$ and $(p, F) \in D_x$. Thus, each D_x is dense in $\mathbb{P}^*$. Hence, since $\mathfrak{c} < \kappa$ and κ is regular, we may assume the each g_α is a total function.

For each $f \in \mathbb{R}^\mathbb{R}$ define

$$D(f) = \{(p, E) \in \mathbb{P}^* : \ f \in E\}.$$

Note that for any (p, F) if we let $E = F \cup \{f\}$, then $(p, F) \leq (p, E)$. Hence $D(f)$ is dense.

Next, note that by the nature of definition of $\leq$ in $\mathbb{P}^*$, if $(p, F) \in G$, where G is a $\mathbb{P}^*$-filter, and $g = \bigcup\{p : \ \exists F \ (p, F) \in G\}$, then for any $f \in F$ we have $g(x) \neq f(x)$ except possibly for the x in the domain of p. Therefore for any $f \in \mathbb{R}^\mathbb{R}$ there exists $\alpha < \kappa$ such that $|g_\alpha \cap f| < \mathfrak{c}$. Thus, the family $\{g_\alpha : \alpha < \kappa\}$ shows that $\mathrm{A}(\mathcal{A}) = \mathfrak{e}_\mathfrak{c} \leq \kappa$ as was to be shown. □

Lemma 3.3 *For any regular κ we have $Lus_\kappa(\mathbb{P}^*) \longrightarrow MA_\kappa(\mathbb{P}^*) \longrightarrow MA_\kappa(\mathbb{P})$.*

PROOF. This first implication needs that κ is regular but is true for any partial order. Given a family $\mathcal{D}$ of dense subsets of $\mathbb{P}^*$ of cardinality less than κ and $\langle G_\alpha : \alpha < \kappa \rangle$ a Lusin sequence for $\mathbb{P}^*$ it must be that for some $\alpha < \kappa$ that G_α meets every element of $\mathcal{D}$.

The second implication follows from the fact that in some sense $\mathbb{P}$ is 'living inside' of $\mathbb{P}^*$. Let $r : \mathbb{R} \to \mathbb{R}$ be a map with of $|r^{-1}(y)| = \mathfrak{c}$ for every $y \in \mathbb{R}$. Define

$$\pi : \mathbb{P}^* \to \mathbb{P} \text{ by } \pi(p, F) = r \circ p.$$

Notice that if $(p, E) \leq (q, F)$ then $\pi(p, E) \leq \pi(q, F)$. This implies that $\pi(G)$ is a $\mathbb{P}$-filter for any $\mathbb{P}^*$-filter G. Furthermore, we claim that if $D \subseteq \mathbb{P}$ is dense, then $\pi^{-1}(D)$ is dense in $\mathbb{P}^*$. To see this, let $(p, F) \in \mathbb{P}^*$ be arbitrary. Since D is dense, there exists $q \leq \pi(p, F)$ with $q \in D$. Now, find $s \in \mathbb{P}$ extending p such that $r \circ s = q \supseteq r \circ p$ and $s(x) \neq f(x)$ for every $x \in \mathrm{dom}(s) \setminus \mathrm{dom}(p)$ and $f \in F$. This can be done by choosing

$$s(x) \in r^{-1}(q(x)) \setminus \{f(x) : f \in F\}$$

for every $x \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$. Then, $(s, F) \leq (p, F)$ and $(s, F) \in \pi^{-1}(q) \subseteq \pi^{-1}(D)$.

This gives us the second implication, since if $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and G is a $\mathbb{P}^*$-filter meeting each element of $\{\pi^{-1}(D) : D \in \mathcal{D}\}$, then $\pi(G)$ is a $\mathbb{P}$-filter meeting each element of $\mathcal{D}$. □

It follows from Lemmas 3.1, 3.2, and 3.3 that $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ implies $\mathrm{A}(\mathcal{A}) = \kappa$. In particular, Theorem 3.1 follows from the following theorem.

Theorem 3.4 *Let $\lambda \geq \kappa \geq \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$ and $Lus_\kappa(\mathbb{P}^*)$ holds.*

Theorem 3.4 will be proved in Section 4.

4 Consistency of our forcing axioms.

In this section we will prove Theorems 3.3 and 3.4. For Theorem 3.3, start with a model of GCH and extend it by forcing with the countable partial functions from λ to ω_1. For Theorem 3.4 start with a model of

$$2^\omega = \omega_1 + 2^{\omega_1} = \lambda$$

and do a countable support iteration of $\mathbb{P}^*$ of length κ. $\mathbb{P}^*$ is isomorphic to the eventual dominating partial order. For the expert this should suffice. The rest of this section is included for our readers who are not set theorists. For similar proofs see for example Kamo [5] and Uchida [16].

We begin with some basic forcing terminology and facts. (See Kunen [6].) For a model M of set theory ZFC and a partial order set $(\mathbb{S}, \leq)$ a filter $G \subseteq \mathbb{S}$ is $\mathbb{S}$-*generic over* M if G intersects every dense $D \subseteq \mathbb{S}$ belonging to M. The fundamental theorem of forcing states that for every model M of ZFC and every partial order $\mathbb{S}$ from M there exists model $M[G]$ of ZFC (called an $\mathbb{S}$-*generic extension of* M) such that G is $\mathbb{S}$-generic over M and $M[G]$ is the smallest model of ZFC such that $M \subseteq M[G]$ and $G \in M[G]$. Thus, the simplistic idea for getting $\mathrm{MA}_\kappa(\mathbb{P})$ is to start with model M of ZFC, take $\mathbb{P}$ from M and look at the model $M[G]$, where G is $\mathbb{P}$-generic over M. Then, G intersects "all" dense subsets of $\mathbb{P}$ and we are done. There are, however, two problems with this simple approach. First, "all" dense subsets of $\mathbb{P}$ means "all dense subsets from M" and we like to be able to talk about all dense subsets from our universe, i.e., from $M[G]$. Second, our partial order is a set described by some formula as the set having some properties. There is no reason, in general, that the same description will give us the same objects in M and in $M[G]$.

The second problem will not give us much trouble. For the generic extensions we will consider, the definition of $\mathbb{P}$ will give us the same objects in all models we will consider. In the case of the partial order $\mathbb{P}^*$ this will not be the case, but the new orders $\mathbb{P}^*$ will be close enough to the old so that it will not bother us.

To take care of the first of the mentioned problems, we will be constructing a Lusin sequence $\langle G_\alpha : \alpha < \kappa \rangle$ by some kind of induction on $\alpha < \kappa$: our final model can be imagined as $N = M[G_0][G_1]\ldots[G_\alpha]\ldots$ and we will make sure that every dense subset $D \in N$ of $\mathbb{P}^*$ is taken care of from some stage $\alpha < \kappa$.

We need some more definitions and facts. Given a partial order we say that p, q are *compatible* if there exists r such that $r \leq p$ and $r \leq q$. A partial order is *well-met* provided for any two elements p, q if p and q are compatible, then they have a greatest lower bound, i.e., there exists r such that $r \leq p$ and $r \leq q$ and for any s if $s \leq p$ and $s \leq q$, then $s \leq r$. Notice that both partial orders $\mathbb{P}$ and $\mathbb{P}^*$ used in Lemmas 3.1 and 3.2 are well-met. For the case of $\mathbb{P}^*$ if (p, E) and (q, F) are compatible, then $(p \cup q, E \cup F)$ is there greatest lower bound. A subset L of a partial order is *linked* if any two elements of L are compatible. A partial order is ω_1-*linked* provided it is a union of ω_1 linked subsets. Assuming the Continuum Hypothesis note that the poset $\mathbb{P}$ used in the proof of Lemma 3.1 has cardinality ω_1 hence it is ω_1-linked. Note that for any $p \in \mathbb{P}$ if we define

$$L_p = \{(q, F) \in \mathbb{P}^* : q = p\},$$

then L_p is a linked subset of $\mathbb{P}^*$, hence $\mathbb{P}^*$ is also ω_1-linked. A subset A of a partial order is an *antichain* if any two elements of A are incompatible. We say that a partial order has the ω_2-*chain condition* (ω_2-*cc*) if every its antichain has cardinality less than ω_2. Clearly ω_1-linked implies the ω_2-chain condition. Finally we say a partial order is countably closed if any descending ω-sequence

$\langle p_n : n \in \omega \rangle$ (i.e., $p_{n+1} \leq p_n$ all n) has a lower bound. Notice that both of our partial orders are countably closed.

All partial orders we are going to consider here will be countably closed and will satisfy ω_2-chain condition. In particular, it is known that if the generic extension $M[G]$ of M is obtained with such partial order, then $M[G]$ and M have the same cardinal numbers, the same real numbers, the same countable subsets of real numbers and the same sets $\mathbb{R}^X$ for any countable set $X \in M$. In particular, $\mathbb{P}$ will be the same in $M[G]$ as in M.

Let us also notice that every dense set contains a maximal antichain and if A is a maximal antichain, then $D = \{p : \exists q \in A \ p \leq q\}$ is a dense set. Thus a filter G is $\mathbb{S}$-generic over a model M if and only if it meets every maximal antichain in M.

PROOF OF THEOREM 3.3. Take a model M of ZFC+GCH. For a set X in M let

$$\mathbb{S}_X = \{p \in \mathbb{P}^X : p(x) = \emptyset \text{ for all but countably many } x \in X\}.$$

Define an ordering on $\mathbb{S}_X$ by $p \leq q$ if and only if $p(x) \leq q(x)$ for every $x \in X$.

Now, let λ be as in Theorem 3.3 and let G be a $\mathbb{S}_\lambda$ generic over M. We will show that $\mathrm{MA}_\lambda(\mathbb{P})$ holds in $M[G]$.

It is easy to see that $\mathbb{S}_\lambda$ is countably closed. It is also known that $\mathbb{S}_\lambda$ satisfies ω_2-cc and that $2^{\omega_1} = \lambda$ in $M[G]$. (See Kunen [6, Ch. VII, Lemma 6.10 and Thm. 6.17].)

Now, for $\alpha < \lambda$ let $G_\alpha = \{p(\alpha) : p \in G\}$. Then, each G_α is a filter in $\mathbb{P}$. We will show that for every family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \lambda$ there exists $\alpha < \kappa$ such that G_α intersects every D from $\mathcal{D}$.

In order to argue for it we need two more facts about forcing $\mathbb{S}_X$. (See Kunen [6, Ch. VII]: Thm. 1.4 and 2.1 for (A) and Lemma 5.6 for (B).)

(A) If $X, Y \in M$ are disjoint and G is $\mathbb{S}_{X\cup Y}$-generic over M, then $G_X = G \cap \mathbb{S}_X$ is $\mathbb{S}_X$-generic over M, G_Y is $\mathbb{S}_Y$-generic over $M[G_X]$, and

$$M[G_X][G_Y] = M[G].$$

(B) If $A \subseteq M$ then there exists $X \in M$ with $|X| \leq |A| + \omega_1$ such that $A \in M[G_X]$.

Now, let G_λ be $\mathbb{S}_\lambda$ generic over M and let $\mathcal{D} \in M[G_\lambda]$ be a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \lambda$. Let $\mathcal{H}$ be a family of maximal antichains, one contained in each element of $\mathcal{D}$. Then, $|A| \leq \omega_1$ for each $A \in \mathcal{H}$, since $\mathbb{P}$ satisfies ω_2-cc. So, by (B), there is $X \subseteq \lambda$ from M of cardinality $|\mathcal{H}| \cdot \omega_1 < \lambda$

such that $\mathcal{H} \in M[G_X]$. Choose $\alpha \in \lambda \setminus X$. Then since G_α is $\mathbb{P}$-generic over $M[G_X]$ it follows that G meets each element of $\mathcal{H}$ hence of $\mathcal{D}$. □

Next we prepare to prove Theorem 3.4. As mentioned in the beginning of the section, we will try to prove it by defining some sequence $\langle \mathbb{S}_\alpha : \alpha \leq \kappa \rangle$ of partial orders and try to obtain our final model as $N_\kappa = M[G_\kappa]$ where every G_α is an $\mathbb{S}_\alpha$-generic over an appropriate initial model. This technique is called iterated forcing and needs a few words of introduction.

We can define in M an iterated forcing $\langle \mathbb{S}_\alpha : \alpha < \kappa \rangle$ by induction on α. At successor stages we define

$$\mathbb{S}_{\alpha+1} = \mathbb{S}_\alpha \times \mathbb{P}^{*M[G_\alpha]}.$$

where $\mathbb{P}^{*M[G_\alpha]}$ is $\mathbb{P}^*$ in the sense of $M[G_\alpha]$. (Since we add new elements of $\mathbb{R}^\mathbb{R}$ the partial order $\mathbb{P}^*$ changes as our models increase.) We can't really do it precisely this way, because $\mathbb{P}_{\alpha+1}$ must be in M. However, it is possible to find its approximation, $\hat{\mathbb{P}}_\alpha$, in M, called a name for $\mathbb{P}_\alpha$, and use this instead. (See Kunen [6, Ch. VII sec. 5]).

For limit ordinals $\lambda < \kappa$, define $\mathbb{S}_\lambda$ to a set of functions f with domain λ such that $f|_\alpha \in \mathbb{S}_\alpha$ for each $\alpha < \lambda$ and $f(\alpha) = \mathbb{I}$ for all but countable many α. Here we use $\mathbb{I}$ to denote the largest element of any partial order. Countable support iterations originated with Laver [8]. For details see Baumgartner [1] or Kunen [6, Ch. VII sec. 7].

The proof that follows will involve a basic lemma used to show various generalizations of Martin's Axiom hold for one cardinal up. (See Baumgartner [1] and Shelah [13]). In particular, we will need the following theorem.

Theorem 4.1 *(Baumgartner) Assume the Continuum Hypothesis. Suppose $\langle \mathbb{S}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of countably closed well-met ω_1-linked partial orders. Then for every $\alpha \leq \kappa$ we have that $\mathbb{S}_\alpha$ is countably closed and satisfies the ω_2-chain condition.*

Actually we need only a very weak version of this theorem, for example, something analogous to [6, Theorem VII, 7.3] of Kunen.

Now, we are ready for the proof of Theorem 3.4.

PROOF OF THEOREM 3.4. Take a model M of ZFC+CH in which $2^\mathfrak{c} = \lambda$, and κ is a regular cardinal with $\omega_2 \leq \kappa \leq \lambda$. Let $\mathbb{S}_\alpha$ be a countable support iteration $\{\mathbb{P}_\alpha : \alpha < \kappa\}$, where $\mathbb{P}_\alpha = \mathbb{P}^{*M[\overline{G^\alpha}]}$ for all $\alpha < \kappa$. Here for $\alpha < \kappa$ let $G^\alpha = G^\kappa|_\alpha$. Then G^α is $\mathbb{S}_\alpha$-generic filter over M.

Let G^κ be an $\mathbb{S}_\kappa$-generic filter over M. We will show that $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ holds in $M[G^\kappa]$.

In the model $M[G^\alpha]$ the partial order $\mathbb{P}^{*M[G^\alpha]}$ can be decoded from $\mathbb{S}_\alpha$ and we can also decode a filter G_α which is $\mathbb{P}^{*M[G^\alpha]}$-generic over $M[G^\alpha]$. We claim that the sequence $\langle G_\alpha : \alpha \in \kappa \rangle$ is a Lusin sequence for $\mathbb{P}^*$ in $M[G^\kappa]$.

So, let $D \in M[G^\kappa]$ be a dense subset of $\mathbb{P}^*$ and let $A \in M[G]$ be a maximal antichain contained in $D \subseteq \mathbb{P}^*$. Then, $|A| \leq \omega_1$, since $\mathbb{P}^*$ satisfies ω_2-cc. So, by the fact similar to (B) above, there is $\beta < \kappa$ such that $A \in M[G^\beta]$. Then, for every $\alpha \geq \beta$, the filter G_α is generic over $M[G^\alpha] \supseteq M[G^\beta]$ and so, G_α intersects both A and D. Therefore, the set

$$\{\alpha < \kappa : G_\alpha \cap D = \emptyset\} = \{\alpha < \kappa : G_\alpha \cap A = \emptyset\} \subseteq \beta$$

has cardinality less then κ. □

It is worth mentioning that some generalizations of the these theorems are possible where the Continuum Hypothesis fails.

References

[1] J.Baumgartner, *Iterated forcing*, in **Surveys in Set Theory** (edited by A.R.D.Mathias), London Mathematical Society Lecture 87, Cambridge University Press, 1983, 1–59.

[2] E.van Douwen and W.Fleissner, Definable forcing axiom: an alternative to Martin's axiom, *Topology Appl. 35* (1990), 277-289.

[3] H.Fast, Une remarque sur la propriété de Weierstrass, *Colloquium Mathematicum 7* (1959), 75-77.

[4] Z.Grande, A.Maliszewski and T. Natkaniec, Some problems concerning almost continuous functions, Proceedings of the 1994 Łódź Summer Workshop, *Real Analysis Exchange*, this volume.

[5] S.Kamo, Some statement which implies the existence of Ramsey ultrafilters on ω, *Journal of the Mathematical Society of Japan 35* (1983), 331-343.

[6] K.Kunen, *Set Theory, an introduction to independence proofs*, North-Holland, 1980.

[7] A.Landver, Baire numbers, uncountable Cohen sets, and perfect-set forcing, *Journal of Symbolic Logic 57* (1992), 1086-1107.

[8] R.Laver, On the consistency of Borel's conjecture, *Acta Mathematica 137* (1976), 151-169.

[9] A.Miller, *Special sets of reals*, in **Set Theory of the Reals** (edited by Haim Judah), Israel Mathematical Conference Proceedings, 6(1993), 415-432, American Math Society.

[10] A.Miller, K.Prikry, When the continuum has cofinality ω_1, *Pacific Journal of Mathematics 115* (1984), 399-407.

[11] T.Natkaniec, Almost continuity, *Real Analysis Exchange 17* (1991–92), 462–520.

[12] T.Natkaniec, *Almost Continuity*, Bydgoszcz 1992.

[13] S.Shelah, A weak generalization of MA to higher cardinals, *Israel Journal of Mathematics 30* (1978), 297-306.

[14] J.Stallings, Fixed point theorem for connectivity maps, *Fund Math. 47* (1959), 249–263.

[15] S.Todorcevic, Remarks on Martin's axiom and the continuum hypothesis, *Canadian Journal of Mathematics 43* (1991), 832-851.

[16] Y.Uchida, Scales on $\omega_1^{\omega_1}$, *Mathematica Japonica 29* (1984), 621-630.