Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 651–656

Ewa Lazarow[1] Department of Mathematics, University of Łódź, Banacha 22, 90-238 Łódź, Poland, e-mail: elazar@plunlo51.bitnet

Aleksander Maliszewski[2] Department of Mathematics, Pedagogical University, Chodkiewicza 30, 85-064 Bydgoszcz, Poland, e-mail: wspb05@pltumk11.bitnet

DECOMPOSITION OF $\mathcal{I}$-APPROXIMATE DERIVATIVES

Abstract

It is shown that if $f\colon \mathbb{R} \to \mathbb{R}$ has a finite $\mathcal{I}$-approximate derivative $f'_{\mathcal{I}\text{-ap}}$ everywhere in $\mathbb{R}$, then there is a sequence of perfect sets, H_n, whose union is $\mathbb{R}$, and a sequence of differentiable functions, h_n, such that $h_n = f$ over H_n and $h'_n = f'_{\mathcal{I}\text{-ap}}$ over H_n. This result is a complete analogue of that on approximately differentiable functions by R. J. O'Malley.

The notion of $\mathcal{I}$-approximate differentiation [2] is based upon the notion of an $\mathcal{I}$-density point which was introduced in [5], and which further properties were studied in [6].

Throughout this paper, $\mathcal{B}$ will denote the family of all subsets of $\mathbb{R}$ (the real line) which possess the Baire property, and $\mathcal{I}$ will denote the σ-ideal of all meager sets.

We say that 0 is an *$\mathcal{I}$-density point* of a set $A \in \mathcal{B}$ iff for every increasing sequence (n_m) of positive integers there exists a subsequence (n_{m_p}) such that

$$\chi_{(n_{m_p}\cdot A)\cap[-1,1]} \longrightarrow 1 \qquad \text{as } p \to \infty$$

except for a meager set. (The symbol $n_{m_p} \cdot A$ stands for $\{n_{m_p} \cdot y : y \in A\}$.) A point x_0 is an $\mathcal{I}$-density point of a set $A \in \mathcal{B}$ iff 0 is an $\mathcal{I}$-density point of $-x_0 + A$. (Similarly, $-x_0 + A = \{-x_0 + y : y \in A\}$.)

Key Words: points of $\mathcal{I}$-density, $\mathcal{I}$-approximate derivative
Mathematical Reviews subject classification: Primary 26A24. Secondary 26A21
Received by the editors October 10, 1994
[1]Supported by a KBN Research Grant 691 02 91, 1991–94
[2]Supported by a KBN Research Grant 2 1144 91 01, 1992–94.

Let f be any finite function defined in some neighborhood of x_0, and having there the Baire property. We define the *$\mathcal{I}$-approximative upper derivate* as the greatest lower bound of the set

$$\left\{\alpha \in \mathbb{R} : \left\{x : \frac{f(x)-f(x_0)}{x-x_0} < \alpha\right\} \text{ has } x_0 \text{ as an } \mathcal{I}\text{-density point}\right\}.$$

The *$\mathcal{I}$-approximate lower derivate* is defined similarly. If the two derivates coincide, their common value is called the *$\mathcal{I}$-approximate derivative* of f at x_0 and denoted by $f'_{\mathcal{I}\text{-ap}}(x_0)$.

One can easily show that if f is $\mathcal{I}$-approximately differentiable at x_0, then x_0 is an $\mathcal{I}$-density point of the set $\{y : |(f(y)-f(x_0))/(y-x_0)-f'_{\mathcal{I}\text{-ap}}(x_0)| < \varepsilon\}$ for each $\varepsilon > 0$.

During the I[st] Joint US-Polish Workshop, Łódź 1994, M. Evans asked a question whether the decomposition theorem holds for $\mathcal{I}$-approximately differentiable functions. We will show it does.

For brevity, we introduce the notation $A \sqsubset B$ for $A \setminus B \in \mathcal{I}$. Moreover, for all $x \in \mathbb{R}$, $h > 0$, $n, k \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$ we define

$$I^+_{ni}(x,h) = \left(x + \frac{i-1}{n}h, x + \frac{i}{n}h\right), \quad I^-_{ni}(x,h) = I^+_{ni}(x-h,h),$$

$$J^+_{nkij}(x,h) = \left(x + \frac{(i-1)k+j-1}{nk}h, x + \frac{(i-1)k+j}{nk}h\right)$$

and $J^-_{nkij}(x,h) = J^+_{nkij}(x-h,h)$.

We will prove the assertion of our main result in two stages. Before the proof we state two lemmas. The first one is an easy consequence of Theorem 1 of [1], and the other is purely technical.

Lemma 1 *Suppose $A \in \mathcal{B}$ and x is an $\mathcal{I}$-density point of A. Then for every $n \in \mathbb{N}$ there are $k, p \in \mathbb{N}$ such that for each $h \in (0, p^{-1})$ and each $i \in \{1, \ldots, n\}$ we can find $j_1, j_2 \in \{1, \ldots, k\}$ with $J^+_{nkij_1}(x,h) \cup J^-_{nkij_2}(x,h) \sqsubset A$.* □

Lemma 2 *Let f be a function with the Baire property which restriction to some closed set K is continuous, and let $m \in \mathbb{N}$. For each $x \in \mathbb{R}$ define*

$$A_m(x) = \{y : |f(y) - f(x)| \le m|y-x|\}.$$

Then for each open interval G the set $B = \{x \in K : x + G \sqsubset A_m(x)\}$ is closed.

PROOF. Suppose there is a sequence (x_n) in B converging to some $x \notin B$, i.e., $(x+G) \setminus A_m(x) \notin \mathcal{I}$. Since $A_m(x) \in \mathcal{B}$, there is a non-degenerate closed interval $U \subset (x+G)$ such that $|f(y) - f(x)| > m|y-x|$ for $\mathcal{I}$-almost every $y \in U$. On the other hand, since $U \subset (x_n + G) \sqsubset A_m(x_n)$ for sufficiently large n, so for $\mathcal{I}$-almost every $y \in U$

$$|f(y) - f(x)| = \lim_{n\to\infty} |f(y) - f(x_n)| \le \lim_{n\to\infty} m|y - x_n| = m|y-x|,$$

contrary to the previous inequality. Hence $x \in B$ and B is closed. □

Theorem 1 *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has a finite $\mathcal{I}$-approximate derivative $f'_{\mathcal{I}\text{-}ap}$ everywhere in $\mathbb{R}$. Then $\mathbb{R}$ can be expressed as the union of a countable family of closed sets, $\mathbf{E}$, such that for each $E \in \mathbf{E}$ and each $x \in E$,*

$$\lim_{y\to x,\, y\in E} \frac{f(y) - f(x)}{y - x} = f'_{\mathcal{I}\text{-}ap}(x). \tag{1}$$

(At an isolated point of E the conclusion is considered to hold vacuously.)

PROOF. It is well known, that $\mathcal{I}$-approximately differentiable functions are Baire * 1 [2, Theorem 3]. So there exists a family of closed sets, $\{K_l : l \in \mathbb{N}\}$, such that $\bigcup_{l\in\mathbb{N}} K_l = \mathbb{R}$ and for each $l \in \mathbb{N}$ the restriction $f|K_l$ is continuous.

For each $m \in \mathbb{N}$ and each $x \in \mathbb{R}$ define $A_m(x)$ as in Lemma 2. Moreover, for all $m, n, k \in \mathbb{N}$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$ and $h > 0$ define

$$C^+_{mni}(h) = \{x : I^+_{ni}(x,h) \sqsubset A_m(x)\},\quad D^+_{mnkij}(h) = \{x : J^+_{nkij}(x,h) \sqsubset A_m(x)\},$$
$$C^-_{mni}(h) = \{x : I^-_{ni}(x,h) \sqsubset A_m(x)\},\quad D^-_{mnkij}(h) = \{x : J^-_{nkij}(x,h) \sqsubset A_m(x)\},$$

and for each $p \in \mathbb{N}$ let E_{mnkp} be the set

$$\bigcap_{h\in(0,p^{-1})} \left[\bigcup_{i=1}^{n} C^+_{mni}(h) \cap \bigcup_{i=1}^{n} C^-_{mni}(h) \cap \bigcap_{i=1}^{n} \Big(\bigcup_{j=1}^{k} D^+_{mnkij}(h) \cap \bigcup_{j=1}^{k} D^-_{mnkij}(h) \Big) \right].$$

We will prove that the family $\mathbf{E} = \{E_{mnkp} \cap K_l : m, n, k, p, l \in \mathbb{N}\}$ fulfills the requirements of the theorem.

First observe that by Lemma 2, each set $E_{mnkp} \cap K_l$ is closed. We will show that $\bigcup \mathbf{E} = \mathbb{R}$. Take an $x \in \mathbb{R}$ and an $m > |f'_{\mathcal{I}\text{-ap}}(x)|$. Since f is $\mathcal{I}$-approximately differentiable at x, x is an $\mathcal{I}$-density point of $A_m(x)$. Now use Lemma 1 to find $n, p_1 \in \mathbb{N}$ such that for each $h \in (0, p_1^{-1})$ there are $i_1, i_2 \in \{1, \dots, n\}$ with $I^+_{ni_1}(x,h) \cup I^-_{ni_2}(x,h) \sqsubset A_m(x)$. Using Lemma 1 again, we can find $k, p_2 \in \mathbb{N}$ such that for each $h \in (0, p_2^{-1})$ and each $i \in \{1, \dots, n\}$ there are $j_1, j_2 \in \{1, \dots, k\}$ for which $J^+_{nkij_1}(x,h) \cup J^-_{nkij_2}(x,h) \sqsubset A_m(x)$. Set $p = \max\{p_1, p_2\}$. Then evidently $x \in E_{mnkp}$. Since $\bigcup_{l\in\mathbb{N}} K_l = \mathbb{R}$, we get $\bigcup \mathbf{E} = \mathbb{R}$.

It remains only to show that f is differentiable relative to each $E_{mnkp} \cap K_l$ and differentiates to $f'_{\mathcal{I}\text{-ap}}$. Fix an $E = E_{mnkp} \cap K_l \in \mathbf{E}$ and an $x \in E$. Let (x_r) be a sequence of points of E converging to x. It will not hurt the generality of the argument to assume that $x = 0$, $f(0) = 0$ and $x_r \searrow 0$.

Note that for each $y_1, y_2 \in E$ with $0 < |y_1 - y_2| < p^{-1}$ we have

$$|f(y_2) - f(y_1)| \le m|y_2 - y_1|. \tag{2}$$

Indeed, let $y_1 < y_2$. By the definition of the set E, there is an $i \in \{1, \dots, n\}$ with $y_1 \in C^{+}_{mni}(y_2 - y_1)$. Further there exists a $j \in \{1, \dots, k\}$ such that $y_2 \in D^{-}_{mnkij}(y_2 - y_1)$. Hence we can find a $z \in A_m(y_2) \cap A_m(y_1) \cap (y_1, y_2)$. So

$$\begin{aligned} |f(y_2) - f(y_1)| &\le |f(y_2) - f(z)| + |f(z) - f(y_1)| \\ &\le m|y_2 - z| + m|z - y_1| = m|y_2 - y_1|. \end{aligned}$$

Now fix an arbitrary $\varepsilon > 0$. Set $V = \{y : |f(y)/y - f'_{\mathcal{I}\text{-ap}}(0)| < \varepsilon\}$. Then 0 is an $\mathcal{I}$-density point of V.

Take an arbitrary integer $s > 1/\varepsilon$ and put $t = ns$. By Lemma 1, we can find a $p_t > p$ such that for each $h \in (0, p_t^{-1})$ and each $j \in \{1, \dots, t\}$,

$$I^{+}_{tj}(0, h) \cap V \notin \mathcal{I}. \tag{3}$$

Let r_0 be such that $x_r < p_t^{-1}$ for $r > r_0$. Since $x_r \in E_{mnkp}$, for each such r there exists an $i \in \{1, \dots, n\}$ with $I^{-}_{ni}(x_r, x_r/s) \sqsubset A_m(x_r)$. So by (3), we can choose a $y_r \in V \cap A_m(x_r) \cap ((1 - \varepsilon)x_r, x_r)$. Now

$$\begin{aligned} \limsup_{r\to\infty} \left| \frac{f(x_r)}{x_r} - f'_{\mathcal{I}\text{-ap}}(0) \right| &\le \limsup_{r\to\infty} \left| \frac{f(x_r) - f(y_r)}{x_r - y_r} \right| \cdot \limsup_{r\to\infty} \left(1 - \frac{y_r}{x_r}\right) \\ &\quad + \limsup_{r\to\infty} \left| \frac{f(y_r)}{y_r} - f'_{\mathcal{I}\text{-ap}}(0) \right| \cdot \frac{y_r}{x_r} + |f'_{\mathcal{I}\text{-ap}}(0)| \cdot \limsup_{r\to\infty} \left(1 - \frac{y_r}{x_r}\right) \\ &\le m \cdot \varepsilon + \varepsilon + |f'_{\mathcal{I}\text{-ap}}(0)| \cdot \varepsilon = (m + 1 + |f'_{\mathcal{I}\text{-ap}}(0)|) \cdot \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrary, the above inequality completes the proof. □

The theorem below is the so-called decomposition theorem. (The sequence (h_n, H_n) is called a decomposition of f. The corresponding sequence (h'_n, H_n) is a decomposition of $f'_{\mathcal{I}\text{-ap}}$.) It is completely analogous to that for approximately differentiable functions by R. J. O'Malley [3, Theorem 2].

Theorem 2 *If $f\colon \mathbb{R} \to \mathbb{R}$ has a finite $\mathcal{I}$-approximate derivative, $f'_{\mathcal{I}\text{-ap}}$, at every point of $\mathbb{R}$, then there is a sequence of perfect sets, H_n, and a sequence of differentiable functions, h_n, such that* *(i)* $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{R}$,

(ii) $h_n(x) = f(x)$ *over* H_n, *and*

(iii) $h'_n(x) = f'_{\mathcal{I}\text{-}ap}(x)$ *over* H_n.

PROOF. We will first obtain sets H_n. Let E_n be the sets defined in Theorem 1. Each closed set can be written as the sum of a countable family of non-degenerate closed intervals, a countable family of nowhere dense perfect sets and a countable set, so we may assume that each E_n is either a non-degenerate closed interval, a nowhere dense perfect set or a singleton.

If E_n is a non-degenerate closed interval or a nowhere dense perfect set, then we set $H_n = E_n$. Since by (1), f is differentiable over H_n, we are able to apply the theorem of Petruska and Laczkovich [4]. This theorem guarantees the existence of a differentiable function, h_n, such that $h_n = f$ over H_n. Then by (1), $h'_n = f'_{\mathcal{I}\text{-ap}}$ over H_n.

So assume that E_n is a singleton. We will be done if we construct a nowhere dense perfect set $H_n \supset E_n$ such that

$$\lim_{y\to x,\, y\in H_n} \frac{f(y)-f(x)}{y-x} = f'_{\mathcal{I}\text{-ap}}(x) \tag{4}$$

holds for each $x \in H_n$.

Let $E_n = \{x\}$. It will not hurt the generality of the argument to assume that $x = 0$ and $f(0) = 0$. Set $\lambda_0 = \infty$. We will construct by induction two sequences of non-empty nowhere dense perfect sets, P_k and Q_k, and a sequence of positive numbers, λ_k, such that the following conditions hold:

- $\lambda_k \searrow 0$ as $k \to \infty$,
- for each $k \in \mathbb{N}$ there are $r_k, l_k \in \mathbb{N}$ such that $P_k \subset E_{r_k} \cap (\lambda_k, \lambda_{k-1})$ and $Q_k \subset E_{l_k} \cap (-\lambda_{k-1}, -\lambda_k)$,
- $\left|f(y)/y - f'_{\mathcal{I}\text{-ap}}(0)\right| < k^{-1}$ for each $k \in \mathbb{N}$ and each $y \in P_k \cup Q_k$.

Suppose we have already constructed non-empty nowhere dense perfect sets $P_1, Q_1, \dots, P_{k-1}, Q_{k-1}$ and numbers $\lambda_1 > \dots > \lambda_{k-1} > 0$ which satisfy the above requirements. Set $V = \{y : \left|f(y)/y - f'_{\mathcal{I}\text{-ap}}(0)\right| < k^{-1}\}$. Then 0 is an $\mathcal{I}$-density point of V. By Lemma 1, we can find a $p > \lambda_{k-1}^{-1}$ such that

$$I^+_{22}(0,p^{-1}) \cap V \notin \mathcal{I} \qquad \text{and} \qquad I^-_{21}(0,p^{-1}) \cap V \notin \mathcal{I}.$$

Since $\bigcup_{n\in\mathbb{N}} E_n = \mathbb{R}$, so by Baire Category Theorem, there are $r_k, l_k \in \mathbb{N}$ such that $V \cap (p^{-1}/2, p^{-1}) \cap E_{r_k} \notin \mathcal{I}$ and $V \cap (-p^{-1}, -p^{-1}/2) \cap E_{l_k} \notin \mathcal{I}$. Hence we can find non-empty nowhere dense perfect sets $P_k \subset V \cap E_{r_k} \cap (p^{-1}/2, p^{-1})$

and $Q_k \subset V \cap E_{l_k} \cap (-p^{-1}, -p^{-1}/2)$. Set $\lambda_k = p^{-1}/2$. It is clear that the conditions above are satisfied.

Set $H_n = E_n \cup \bigcup_{k \in \mathbb{N}} (P_k \cup Q_k)$. Then evidently H_n is nowhere dense and perfect, and the condition (4) holds. This completes the proof. □

References

[1] E. Łazarow, *On the Baire class of $\mathcal{I}$-approximate derivatives*, Proc. Amer. Math. Soc. **100** (1987), no. 4, 669–674.

[2] E. Łazarow and W. Wilczyński, *$\mathcal{I}$-approximate derivatives*, Rad. Mat. **5** (1989), 15–27.

[3] R. J. O'Malley, *Decomposition of approximate derivatives*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 243–247.

[4] G. Petruska and M. Laczkovich, *Baire one functions, approximately continuous functions and derivatives*, Acta Math. Hungar. **25** (1974), 189–212.

[5] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *A category analogue of the density topology*, Fund. Math. **125** (1985), 167–173.

[6] W. Wilczyński and V. Aversa, *Some remarks on $\mathcal{I}$-approximately continuous functions*, Ricerche Mat. **33** (1984), no. 1, 63–79.