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SMOOTHING Λ-SEQUENCES

Abstract

It is shown that for every Λ-sequence Λ there is an equivalent Λ-sequence $\Gamma = (\gamma_i)$ such that $\limsup \gamma_{i+1}/\gamma_i = 1$.

In a recent investigation concerning bounded Λ-variation as a gap Tauberian condition, a question about ΛBV spaces arose which has not been previously considered. We quickly recapitulate the essential facts about these spaces. Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n = \infty$. A function f defined on an interval (finite or infinite) is of Λ-bounded variation if $\sum |f(b_n) - f(a_n)|/\lambda_n$ converges for every sequence $\{[a_n, b_n]\}$ of nonoverlapping intervals. The class of such functions is known as ΛBV. It may be shown that such functions are regulated, i.e., right and left limits exist at each point. We generally assume that $\lambda_n \nearrow \infty$, for otherwise, $\Lambda BV = BV$ [W],[A].

In the study of the Tauberian theorem we referred to, it seemed necessary to make the assumption that $\limsup \lambda_{n+1}/\lambda_n < \infty$. A question which arises naturally is

Question 1: Given a class ΛBV for which $\limsup \dfrac{\lambda_{n+1}}{\lambda_n} = \infty$, is there a $\Gamma = \{\gamma_n\}$, with $\limsup \dfrac{\gamma_{n+1}}{\gamma_n} < \infty$, such that $\Gamma BV = \Lambda BV$?

When $\Gamma BV = \Lambda BV$, we shall say that the sequences Λ and Γ are equivalent.

After we answered this question affirmatively, the next to come to mind was the following question.

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Question 2: For any given Λ, is there a Γ equivalent to Λ such that $\lim \dfrac{\gamma_{n+1}}{\gamma_n} = 1$?

A Λ-sequence is called *smooth* if $\dfrac{\lambda_{n+1}}{\lambda_n} \to 1$.

Question 2 was also answered affirmatively. The method employed for Question 1 consisted of altering a subsequence of Λ to form the desired Γ. The method employed for Question 2 is an amplification of the original argument. This method, although direct, is relatively complicated.

Another question, which also has an affirmative answer, is

Question 3: Is there a computationally simple method by which one can obtain a smooth Γ equivalent to a given Λ?

This is an appropriate time to remind the reader of a theorem of Perlman and Waterman [PW]:

Theorem Λ *and* Γ *are equivalent if and only if there are positive constants* c *and* c' *such that, for every* n,

$$c \le \frac{\sum_1^n 1/\gamma_k}{\sum_1^n 1/\lambda_k} \le c'.$$

We will present our solutions for each question since there is a logical progression from one to the next and there is something to be learned about Λ-sequences from each of them. Let us now suppose that $\limsup \dfrac{\lambda_{n+1}}{\lambda_n} = \infty$ and choose a finite $c > 1$. Then $\dfrac{\lambda_{n+1}}{\lambda_n} > c$ for infinitely many values of n. Let $\gamma_1 = \lambda_1$ and, for $n > 1$, let $\gamma_n = \min\{c\gamma_{n-1}, \lambda_n\}$.

Clearly γ_n is nondecreasing and $\gamma_n \le \lambda_n$ for every n. Also, $\gamma_n \nearrow \infty$, since $\gamma_n = \lambda_n$ for infinitely many values of n, for otherwise, there is an integer i such that $\gamma_k = c^{k-i}\lambda_i$ for $k > i$ and $\lambda_k \ge \gamma_k$, implying that $\sum \dfrac{1}{\lambda_k}$ converges.

Now

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{\min\{c\gamma_n, \lambda_{n+1}\}}{\min\{c\gamma_{n-1}, \lambda_n\}} .$$

We have either

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{c\gamma_n}{\gamma_n} = c$$

or

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{\lambda_{n+1}}{\gamma_n} \le \frac{c\gamma_n}{\gamma_n} = c$$

so that $\limsup \dfrac{\gamma_{n+1}}{\gamma_n} \le c$.

Also, since γ_k is either λ_k or $c^{k-i}\lambda_i$ for some i less than k ,

$$1 \le \frac{\sum_1^n 1/\gamma_k}{\sum_1^n 1/\lambda_k} \le \frac{\sum_{i=1}^n (1/\lambda_i \sum_{k=0}^{n-i} 1/c^k)}{\sum_1^n 1/\lambda_i} \le \frac{c}{c-1},$$

implying that Λ and Γ are equivalent.

Question 2 can be dealt with by a modification of this method. Choose $c_n \searrow 1$. Let $\gamma_1 = \lambda_1$. Starting with $n = 2$, we define numbers γ_n successively by $\gamma_n = \min\{\lambda_n, c_1^k\lambda_1\}$, where k is the least integer such that $c_1^k\lambda_1 > \gamma_{n-1}$, until we reach an index $n_1 > 1$ such that $\gamma_{n_1} = \lambda_{n_1}$ and

$$\frac{1}{\lambda_{n_1}} \cdot \frac{1}{c_2 - 1} < \frac{1}{2} .$$

Begin anew to define γ_n for $n > n_1$ by the same method with c_2 replacing c_1 and λ_{n_1} replacing λ_1 until we reach an index $n_2 > n_1$ such that $\gamma_{n_2} = \lambda_{n_2}$ and

$$\frac{1}{\lambda_{n_2}} \cdot \frac{1}{c_3 - 1} < \frac{1}{2^2} .$$

We continue in this way to define $n_k \nearrow$ such that for $n_k < n \le n_{k+1}$ we have

$$\gamma_n = \min\{\lambda_n, c_{k+1}^j\lambda_{n_k}\},$$

where j is the least integer such that $c_{k+1}^j\lambda_{n_k} > \gamma_{n-1}$, $\gamma_{n_{k+1}} = \lambda_{n_{k+1}}$ and

$$\frac{1}{\lambda_{n_{k+1}}} \cdot \frac{1}{c_{k+2} - 1} < \frac{1}{2^{k+1}} .$$

Suppose now that $1 \le n < n_1$ and $\gamma_n = c_1^k\lambda_1$. Then $c_1^k\lambda_1 \le \lambda_n$ and k is the least integer such that $c_1^k\lambda_1 > \gamma_{n-1}$. Then $\gamma_{n+1} = \min\{\lambda_{n+1}, c_1^j\lambda_1\}$, where j is the least integer such that $c_1^j\lambda_1 > \gamma_n = c_1^k\lambda_1$, implying that $j = k+1$ and $\gamma_{n+1}/\gamma_n \le c_1$. If $n < n_1$ and $\gamma_n = \lambda_n < c_1^k\lambda_1$ where k is the least integer such that $c_1^k\lambda_1 > \gamma_{n-1}$, then

$$\frac{\gamma_{n+1}}{\gamma_n} \le \frac{\min\{\lambda_{n+1}, c_1^k\lambda_1\}}{c_1^{k-1}\lambda_1} \le \frac{c_1^k\lambda_1}{c_1^{k-1}\lambda_1} = c_1 .$$

In an analogous fashion for $n_k \le n < n_{k+1}$ we can show that

$$\frac{\gamma_{n+1}}{\gamma_n} \le c_{k+1} .$$

Finally, Γ is equivalent to Λ for

$$1 \le \frac{\sum_1^n 1/\gamma_k}{\sum_1^n 1/\lambda_k}$$

$$\le \frac{\displaystyle\sum_1^n \frac{1}{\lambda_k} + \frac{1}{\lambda_1(c_1-1)} + \sum_1^\infty \frac{1}{\lambda_{n_k}(c_{n_{k+1}}-1)}}{\displaystyle\sum_1^n 1/\lambda_k} = 1 + O\left(1/\sum_1^n \frac{1}{\lambda_k}\right)$$

as $n \to \infty$.

In the above we formed Γ by replacing terms of Λ by smaller values. There is another method, that of interlacing two sequences, which is simpler and furnishes the answer to Question 3.

Note that if there is subsequence $\{\lambda_{n_k}\}$ of Λ such that $\limsup \lambda_{n_{k+1}}/\lambda_{n_k} = 1$, then $\lim \lambda_{n+1}/\lambda_n = 1$, for

$$\lambda_{n_k} \le \lambda_n \le \lambda_{n+1} \le \lambda_{n_{k+1}} \text{ implies } \frac{\lambda_{n+1}}{\lambda_n} \le \frac{\lambda_{n_{k+1}}}{\lambda_{n_k}}.$$

We shall interlace the sequences $\{n^2\}$ and Λ. More precisely, we place those terms of $\{n^2\}$ which are $\le \lambda_1$ before λ_1 and arrange them in increasing order of magnitude. Between λ_1 and λ_2 we place those terms n^2 such that $\lambda_1 < n^2 \le \lambda_2$ arranged in increasing order and we repeat this process for all pairs λ_k and λ_{k+1}. Let γ_k be the k-th term in this sequence. There is a strictly increasing sequence of positive integers $\{n_k\}$ such that $\gamma_{n_k} = \lambda_k$ and we always have $n_k \ge k$. Thus $\sum 1/\gamma_n$ diverges. If $k_n = \min\{k \mid \lambda_k > \gamma_n\}$, then $k_n \le n$, implying $\gamma_n \le \lambda_n$. Then

$$1 \le \frac{\sum_1^n 1/\gamma_k}{\sum_1^n 1/\lambda_k} \le \frac{\sum_1^n 1/\lambda_k + \sum_1^\infty 1/n^2}{\sum_1^n 1/\lambda_k} = 1 + O(1/\sum_1^n \frac{1}{\lambda_k})$$

as $n \to \infty$, so Γ and Λ are equivalent. Choosing $\gamma_{n_k} = 1/k^2$, we have $\lim \gamma_{n_{k+1}}/\gamma_{n_k} = 1$, so $\lim \gamma_{n+1}/\gamma_n = 1$.

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