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A GENERALIZATION OF THE BANACH ZARECKI THEOREM

Abstract

It is well known that the following theorem due to Banach and Zarecki: $AC = VB \cap (N) \cap C$, on a closed set. In [1] we showed that this theorem is no longer true if AC and VB are replaced by Foran's conditions $A(2)$ and $B(2)$, respectively. In the present paper, we introduce the classes AC_∞ and VB_∞, which contain strictly the classes AC and VB, respectively. Then we show that $AC_\infty = VB_\infty \cap (N)$, for bounded measurable functions on a measurable set.

Definition 1 *Let $F : [a, b] \to \mathbb{R}, P \subset [a, b]$. Let $O(F; P) = \sup\{F(y) - F(x) : x, y \in P\}$. Let $O^\infty(F; P) = \inf\{\sum_{i=1}^{\infty} O(F; P_i) : \bigcup_{i=1}^{\infty} P_i = P\}$.*

Clearly, $O^\infty(F; P) \leq O(F; P)$.

Proposition 1 *Let $F : [a, b] \to \mathbb{R}, P \subset [a, b]$. If F is bounded on P then $O^\infty(F; P) = |F(P)|$.*

Proof. We will show that $O^\infty(F; P) \leq |F(P)|$. For $\varepsilon > 0$, there exists an open set G, such that $F(P) \subset G = \cup_{i=1}^{\infty} J_i$, and $|F(P)| + \varepsilon > |G|$, where J_i, $i = 1, 2, \ldots$, are the components of G. Let $P_i = P \cap F^{-1}(J_i)$. Then $F(P) = F(\cup_{i=1}^{\infty} P_i) = \cup_{i=1}^{\infty} F(P_i) \subset \cup_{i=1}^{\infty} J_i$, hence $O(F; P_i) \leq |J_i|$. It follows that $O^\infty(F; P) \leq \sum_{i=1}^{\infty} O(F; P) \leq \sum_{i=1}^{\infty} |J_i| = |G| < |F(P)| + \varepsilon$. Since ε is arbitrary, $O^\infty(F; P) \leq |F(P)|$.

We will show that $|F(P)| \leq O^\infty(F; P)$. For $\varepsilon > 0$ there exists a sequence of sets $\{P_i\}$, $i = 1, 2, \ldots$, such that $P = \cup_{i=1}^{\infty} P_i$, and $O^\infty(F; P) + \varepsilon > \sum_{i=1}^{\infty} O(F; P_i)$. Let $J_i = [\inf(F(P_i)), \sup(F(P_i))]$. Then $F(P) = \cup_{i=1}^{\infty} F(P_i) \subset \cup_{i=1}^{\infty} J_i$, hence $|F(P)| \leq \sum_{i=1}^{\infty} |J_i| = \sum_{i=1}^{\infty} O(F; P_i) < O^\infty(F; P) + \varepsilon$. Since ε is arbitrary, $|F(P)| \leq O^\infty(F; P)$. □

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Definition 2 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$. F is said to be AC_∞ on P, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that $\sum_{k=1}^{p} O^\infty(F; P \cap I_k) < \varepsilon$, whenever $I_k, k = 1, 2, \ldots, p$, is a finite set of nonoverlapping closed intervals with endpoints in P, and $\sum_{k=1}^{p} |I_k| < \delta$.*

Definition 3 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$, F is said to be VB_∞ on P, if there exists a number $M \in (0, +\infty)$, such that $\sum_{k=1}^{p} O^\infty(F; P \cap I_k) < M$, whenever, $\{I_k\}$, is a finite set of nonoverlapping closed intervals, with endpoints in P.*

Proposition 2 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$.*

(i) $AC \subsetneqq AC_\infty$ on P;

(ii) $AC_\infty = AC$ on $[a,b]$, for Darboux functions;

(iii) $VB \subsetneqq VB_\infty$ on P;

(iv) $VB_\infty = VB$ on $[a,b]$, for Darboux functions.

PROOF.

(i) This follows by definitions, and the following example: let f be defined on $[a,b], f(x) = 1$, for $x =$ a rational number, $f(x) = 0$, for $x \neq$ a rational number. Then $f \in AC_\infty$, and $f \notin AC$.

(ii) Let I be a closed subinterval of $[a,b]$. By Proposition 1, $O^\infty(F; I) = |F(I)|$. Since F is Darboux, $F(I)$ is an interval, hence $|F(I)| = O(F; I)$. Hence $O^\infty(F; I) = O(F; I)$. Now the proof follows by definitions.

(iii) and (iv) follow similarly to (i) and (ii). □

Remark 1 *Clearly $AC \subset A(N) \subset AC_\infty$ and $VB \subset B(N) \subset VB_\infty$ on a set P, where $A(N)$ and $B(N)$ are Foran's conditions, introduced in [2].*

Definition 4 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b], F \in VB_\infty$ on P. We denote $V_\infty(F; P) = \inf\{M : M$ is given by the fact that $F \in VB_\infty$ on $P\}$.*

Clearly, $V_\infty(F; P) = \sup\{\sum_{k=1}^{p} |F(P \cap I_k)| : \{I_k\}, k = 1, 2, \ldots, p$, is a finite set of nonoverlapping closed intervals with $I_k \cap P \neq \emptyset\}$.

Definition 5 *Let $P \subset [a,b], F : P \to \mathbb{R}$, and let $s : \mathbb{R} \to \mathbb{R}, s(y) =$ the number of roots of the equation $F(x) = y, x \in P$. $s(y)$ is called the Banach indicatrix. Let $K_p : [a,b] \to \mathbb{R}, K_p(x) = 1, x \in P$, and $K_p(x) = 0, x \notin p$. K_p is called the characteristic function of P.*

Lemma 1 *Let $P \subset [a,b]$ be a measurable set. Let $F : P \to \mathbb{R}$, be a bounded, measurable function, $m = \inf(F(P)), M = \sup(F(P))$. If $F(A)$ is a measurable set whenever A is a measurable subset of P, then:*

(i) $\int_m^M s(y)\,dy = V_\infty(F;P) = \sup\{\sum_{k\geq 1}|F(P_k)| : \{P_K\}$ *is a finite or infinite collection of measurable, pairwise disjoint subsets of P, and* $\cup_{k\geq 1}P_k = P\}$.

(ii) $\Phi(X) = V_\infty(F;X)$ *is an additive set function, where Φ is defined on all measurable subsets X of P, and $V_\infty(F;P) \neq +\infty$.*

PROOF. (i) If $\{P_k\}$ is as above, then we have

$$\sum_{k\geq 1} K_{F(P_k)}(y) \leq s(y), \text{for each } y \in [m,M]. \tag{1}$$

For each natural number $n \geq 1$, let $I_1^n = [a, a+(b-a)/2^n]$, and

$$I_k^n = (a+(k-1)(b-a)/2^n, a+k(b-a)/2^n], n = 2,3,\ldots,2^n.$$

Let $s_n(y) = \sum_{k=1}^{2^n} K_{F(P\cap I_k^n)}(y)$. But $F(P\cap I_k^n)$ is measurable by hypothesis, hence $s_n(y)$ is a positive, measurable function. Clearly $\{s_n(y)\}_n$ is increasing. We show that $s_n(y) \to s(y), n \to \infty$. Let $s^*(y) = \lim_{n\to\infty} s_n(y)$. Then $s^*(y)$ is a positive, measurable function. By [1], $s_n(y) \leq s(y)$, hence $s^*(y) \leq s(y)$. For y let $q(y)$ be a natural number, such that $q(y) \leq s(y)$. Then there exist $q(y)$ distinct roots $x_1 < x_2 < \ldots < x_{q(y)}$, of the equation $F(x) = y, x \in P$. Let $n(y)$ be a natural number, such that $(b-a)/2^{n(y)} < \min\{x_{i+1} - x_i : i = 1,2,\ldots,q(y)-1\}$. Then there exist $k_1 < k_2 < \ldots < k_{q(y)}$, such that $x_i \in P \cap I_{k_i}^{n(y)}, i = 1,2,\ldots,q(y)$. Hence $K_{F(P\cap I_{k_i}^{n(y)})}(y) = 1$. It follows that $s_{n(y)}(y) \geq q(y)$. If $q(y) = s(y) < +\infty$ then $q(y) = s(y) \geq s_{n(y)}(y) \geq q(y)$, hence $s_{n(y)}(y) = s(y) = q(y)$. If $s(y) = +\infty$, then $q(y)$ can be taken arbitrarily large, hence $s^*(y) = +\infty$. It follows that $s(y) = s^*(y)$, and $\lim_{n\to\infty} s_n(y) = s(y)$. By the Beppo-Levi Theorem,

$$\lim_{n\to\infty}\int_m^M s_n(y)\,dy = \int_m^M s(y)\,dy. \text{ By [1]}, \sum_{k\geq 1}\int_m^M K_{F(P_k)}(y)\,dy \geq \int_m^M s(y)\,dy.$$

(ii) Let $\{X_i\}$ be a sequence of measurable, pairwise disjoint sets, $X_i \subset P, i = 1,2,\ldots$. Let $X = \cup_{i=1}^\infty X_i$. Then by (i),

$$\sum_{i=1}^{\infty} V_\infty(F;X_i) = \sum_{i=1}^{\infty} \int_m^M (s_{/X_i})(y)\,dy$$
$$= \int_m^M \sum_{i=1}^{\infty} (s_{/X_i})(y)\,dy = \int_m^M (s_{/X})(y)\,dy = V_\infty(F;X),$$

hence $\Phi(X) = \sum_{i=1}^{\infty} \Phi(X_i)$. □

Corollary 1 *Let P be a measurable set. Let $F : P \to \mathbb{R}$ be bounded measurable function. If F satisfies Lusin's condition (N) on P, then Φ is an additive set function, and Φ is AC on P.*

PROOF. By a theorem of Rademacher [3, p. 354] and Lemma 1, (ii), Φ is additive. Let $X \subset P, |X| = 0$. Let $\{X_k\}, k \geq 1$, be a finite or infinite collection of measurable, pairwise disjoint subsets of X, with $X = \cup_{k\geq 1} X_k$. Since $F \in (N)$ on $P, \sum_{k\geq 1} |F(X_k)| = 0$. By Lemma 1, (i), $\Phi(X) = 0$, hence $\Phi \in AC$ on P, [7, p. 30]. □

If $F : P \to \mathbb{R}$ and $\{P_k\}$ is a finite or infinite collection of pairwise disjoint Borel (resp. analytic) sets with $\cup_{k\geq 1} P_k = P$, we let $V_\infty(F;P) = \sup\{\sum_{k\geq 1}^{\infty} |F(P_k)|\}$.

Corollary 2 *[Iseki, [4, p. 16] and [5, p. 38–39]] Let $P \subset [a,b]$ be a Borel (respectively analytic) set, let $F : P \to \mathbb{R}$. If F is bounded and continuous on P, then the Banach indicatrix, $s(y)$ is measurable and $\int_{\mathbb{R}} s(y)\,dy = V_\infty(F;P)$.*

PROOF. The proof is similar to that of Lemma 1(i), since a continuous image of a Borel (respectively analytic) set is always a measurable set. □

Remark 2 *Lemma 1 and Corollary 2 may be regarded as generalizations of a well known theorem of Banach (see [7, p. 280]). In [4], the integral $\int_m^M s(y)\,dy$ is called the fluctuation of F on the set P.*

Corollary 3 *Let $P \subset [a,b]$, and let $F : P \to \mathbb{R}, F \in VB_\infty$ on P.*

(i) If P is a measurable set, F is a measurable function, and $F \in (N)$ on P then $F \in T_1$ on P.

(ii) If P is a Borel (respectively analytic) set, and F is continuous on P, then $F \in T_1$ on P.

PROOF. (i) follows by Lemma 1 and the definition of Banach's condition T_1 (ii) follows by Corollary 2 and the definition of T_1. □

Definition 6 *Let $P \subset [a,b], F : P \to \mathbb{R}$. F fulfils Banach's condition S on P if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|F(Z)| < \varepsilon$, whenever $Z \subset P$ and $|Z| < \delta$. If in addition Z is supposed to be a compact set, then we obtain condition wS (weak S) on P.*

Definition 7 *Let $P \subset [a,b], F : P \to \mathbb{R}$. F is said to be S_0 on P, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{n} |F(P_i)| < \varepsilon$, whenever P_i, $i = 1,2,...,n$, are measurable, pairwise disjoint subsets of P, with $\sum_{i=1}^{n} |P_i| < \delta$. In addition P_i, $i = 1,2,...,n$, are supposed to be a compact set, we obtain condition wS_0 on P.*

Proposition 3 *Let $F[a,b] \to \mathbb{R}, P \subset [a,b]$. Then we have:*

(i) $AC_\infty \subset S \subset (N)$ *on* P;

(ii) $S_0 \subset S \subset wS$ *on* P;

(iii) $S_0 \subset wS_0 \subset wS$ *on* P;

(iv) *If P is measurable, then* $S = wS \cap (N)$ *on* P;

(v) *If P is of* F_σ *type then* $wS \subset (N)$, *hence* $S = wS$ *on* P.

PROOF.

(i) Let $\varepsilon > 0$, and let δ be given by the fact that $F \in AC_\infty$ on P. Then there exists $\{I_k\}_k$, a sequence of non-overlapping closed intervals, such that $E \subset \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} O^\infty(F; E \cap I_k) < \varepsilon$. By Proposition 1, $|F(E \cap I_k)| = \sum_{k=1}^{\infty} O^\infty(F; E \cap I_k) < \varepsilon$. Hence $F \in S$ on P. For $S \subset (N)$, see [7].

(ii) and (iii) follow from the definitions.

(iv) $S \subset wS \cap (N)$ on P follows by (i) and (ii). Let $F \in wS \cap (N)$ on P. Let $Z \subset P$, Z-measurable, $|Z| < \delta$. We have two situations: 1) Z is a set of F_σ-type. Then there exists $Q_1 \subset Q_2 \subset ... \subset Q_n \subset ...$, compact sets, such that $Z = \cup_{i=1}^{\infty} Q_i$. But $F(Z) = F(\cup_{i=1}^{\infty} Q_i) = \cup_{i=1}^{\infty} F(Q_i)$. Since $\{F(Q_i)\}_i$, is an increasing sequence of sets, it follows that $|F(Z)| = \lim_{n\to\infty} |F(Q_i)|$. But $|Q_i| < \delta$, hence $|F(Q_i)| \le \varepsilon$, $i = 1,2,....$ Then $|F(Z)| \le \varepsilon$, hence $F \in S$ on P. 2) Z is not a set of F_σ-type. Then there exists $A \subset Z$, such that A is a set of F_σ-type, and $|Z - A| = 0$. We have $|F(Z)| \le |F(A)| + |F(Z - A)|$. But $|F(A)| \le \varepsilon$ (see 1), and $|F(Z - A)| = 0$ (since $F \in (N)$). Hence, $|F(Z)| \le \varepsilon$ and $F \in S$ on P.

(v) Let $Z \subset P$, $|Z| = 0$. For $\varepsilon > 0$, let $\delta > 0$ be given by the fact that $F \in wS$ on P. Then there exists an open set Q, such that $Q \supset Z, |Q| < \delta$. It follows that $Z \subset Q \cap P$ and $Q \cap P$ is of F_σ-type. Similarly to (iv) 1, it follows that $|F(Z)| \leq |F(Q \cap P)| < \varepsilon$, hence $|F(Z)| = 0$, and $F \in (N)$ on P.

□

Theorem 1 *Let $F : [a, b] \to \mathbb{R}$, F a bounded and measurable function. Let P be a measurable subset of $[a, b]$. The following assertations are equivalent:*

(1) $F \in AC_\infty$ *on* P*;*

(ii) $F \in wS_0 \cap (N)$ *on* P*;*

(iii) $F \in S_0$ *on* P*;*

(iv) $F \in VB_\infty \cap (N)$ *on* P*.*

PROOF.

(i) $\Rightarrow$ (ii) Let $\varepsilon > 0$, and let δ be given by the fact that $F \in AC_\infty$ on P. Let $\{P_k\}, k = 1, 2, ...n$, be a finite set of pairwise disjoint, compact subsets of P, such that $\sum_{k=1}^{n} |P_k| < \delta/2$. For each P_k, there exists a finite set of non-overlapping closed intervals $I_{k,j}, j = 1, 2, ..., p$, with endpoints in P_k, such that $P_k \subset \cup_{k=1}^{n} \cup_{j=1}^{p} I_{k,j}$ and $\sum_{k=1}^{n} \sum_{j=1}^{p} |I_{k,j}| < \delta$. Then$\sum_{k=1}^{n} |F(P_k)| \leq \sum_{k=1}^{n} \sum_{j=1}^{p} |F(I_{k,j} \cap P)| < \varepsilon$, hence $F \in wS_0$ on P. By proposition 3 (i), $F \in (N)$ on P.

(ii) $\Rightarrow$ (iii). The proof is similar to that of Iseki (see[4], Theorem 14). Let $\varepsilon > 0$. For $\varepsilon/2$, let $\delta > 0$ be given by the fact that $F \in wS$ on P. Let Q be a measurable subset of P. Then there exists a set of F_σ-type A, such that $A \subset Q$ and $|Q - A| = 0$. Since $F \in (N)$ on $P, |F(Q - A)| = 0$. By Proposition 3 (iii), $F \in wS$ on P. Hence $|F(Q)| = |F(A) \cup F(Q - A)| \leq |F(A)| + |F(Q - A)| = |F(A)|$. Since $A \subset Q$, it follows that $|F(Q)| = |F(A)|$. The set A can be expressed as the limit of an increasing, infinite sequence of compact sets. It follows that, for $\varepsilon > 0$ there exists $A_\varepsilon \subset A$, A_ε- a compact set, such that $|F(Q)| = |F(A)| < |F(A_\varepsilon)| + \varepsilon$. Let $\{P_i\}$, $i = 1, 2, ..., n$, be a finite set of measurable, pairwise disjoint subsets of P, such that $\sum_{i=1}^{n} |P_i| < \delta$. Then, as above, there exists a compact set $Q_i \subset P_i$, such that $|F(P_i)| \leq |F(Q_i)| + \varepsilon/2n$, $i = 1, 2, ..., n$. It follows that $\sum_{i=1}^{n} |Q_i| < \delta$ and $\sum_{i=1}^{n} |F(Q_i)| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence $F \in S_0$ on P.

(iii) ⇒(i). For $\varepsilon > 0$, let $\delta > 0$, be given by the fact that $F \in S_0$ on P. Let $\{I_k\}$, $k = 1, 2, ..., n$, be a finite set of nonoverlapping closed intervals, such that $P \cap I_k \neq \phi$, and $\sum_{k=1}^{n} |I_k| < \delta$. Let $P_k = P \cap I_k$. Then $\sum_{k=1}^{n} |P_k|, \delta$, and $\sum_{k=1}^{n} |F(P_k)|, \varepsilon$, hence $F \in AC_\infty$ on P.

(iii) ⇒(iv). Let $F \in S_0$ on P. Then , by Proposition 3 (i),(ii), $F \in (N)$ on P. For $\varepsilon = 1$, let $\delta > 0$, be given by the fact that $F \in S_0$ on P. Let $\{P_k\}$, $k = 1, 2, .., p$ be a finite set of measurable, pairwise disjoint subsets of P, $P = \cup_{k=1}^{p} P_k$ and $\operatorname{diam}(P_k) < \delta$, $k = 1, 2, .., p$. By Lemma 1 (ii),(i), $\Phi(P) = \sum_{k=1}^{p} \Phi(P_k) \leq p$, hence $F \in VB_\infty$ on P.

(iv) ⇒(iii). Let $F \in VB_\infty \cap (N)$ on P. By a well-known theorem of Saks ([7],p. 31), it follows that, for each $\varepsilon > 0$, there exists a $\delta > 0$, such that, for each measurable set $X \subset P$, $\Phi(X) < \varepsilon$, whenever $|X| < \delta$. Let $\{P_k\}$, $k = 1, 2, ..., p$, be a finite collection of measurable, pairwise disjoint subsets of P, with $\sum_{k=1}^{p} |(P_k) < \delta$. Then $\sum_{k=1}^{p} |F(P_k)| \leq |\sum_{k=1}^{p} \Phi(P_k) = \Phi(\cup_{k=1}^{p} P_k) < \varepsilon$, hence $F \in S_0$ on P.

□

Remark 3 *a) From the proof of Theorem 1, it follows that (i) ⇔(ii)⇔(iii) without asking F to be measurable. b) In [1], we showed that there exists a continuous function F, which is B(2) on a perfect set P, F ∈ (N) on P, F ∉ A(N) on P, N = 1, 2, ... That's why (i) ⇔(iv) in Theorem 1 is so surprising.*

Corollary 4 *Let* $F[a, b] \to \mathbb{R}, P \subset [a, b]$.

(i) If P is a set of F_σ*-type then* $AC_\infty = wS_0 = S_0$ *on P;*

(ii) $AC = AC_\infty = S_0 = wS_0$ *on* $[a, b]$, *for continuous functions;*

(iii) $S_0 \subsetneq S \subsetneq (N)$ *and* $wS_0 \subsetneq wS$ *on* $[a, b]$, *for continuous functions; wS on* $[a, b]$, *for continuous functions;*

(iv) $VB \cap (N) = VB_\infty \cap (N) = AC_\infty = AC$ *on* $[a, b]$, *for Darboux functions.*

PROOF. (i) See Proposition 3(iii), (iv) and Theorem 1(i), (ii), (iii). (ii) See (i) and Proposition 2(ii). (iii) By [7] $AC \subsetneq S$, for continuous functions on $[a, b]$. Now the proof follows by (ii) and Proposition 3(ii), (iii), (i). (iv) See Proposition 2(ii), (iv) and Theorem 1. □

Remark 4 *Corollary 4(iv) is in fact the Banach-Zarecki theorem ([7]).*

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