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ON THE MAXIMAL FAMILIES FOR THE CLASS OF STRONGLY QUASI–CONTINUOUS FUNCTIONS

Abstract

It is investigated the maximal families (additive, multiplicative, lattice and with respect to the composition) for the class of strongly quasicontinuous functions.

Let $\mathbb{R}$ be the set of all reals and let μ_e (μ) denote the outer Lebesgue measure (the Lebesgue measure) in $\mathbb{R}$. Denote by

$$d_u(A, x) = \limsup_{h \to 0} \mu_e(A \cap (x-h, x+h))/2h$$

$$(d_l(A, x) = \liminf_{h \to 0} \mu_e(A \cap (x-h, x+h))/2h)$$

the upper (lower) density of a set $A \subset \mathbb{R}$ at a point x. A point $x \in \mathbb{R}$ is called a density point of a set $A \subset \mathbb{R}$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B, x) = 1$. The family $\mathcal{T}_d = \{A \subset \mathbb{R}; A \text{ is measurable and every point } x \in A \text{ is a density point of } A\}$ is a topology called the density topology [1]. Denote by $int(A)$ the interior of the set A. The family

$$\mathcal{T}_{ae} = \{A \in \mathcal{T}_d; \mu(A - int(A)) = 0\}$$

is also a topology [4].

A function f (from $\mathbb{R}$ into $\mathbb{R}$) is called $\mathcal{T}_{ae}-$ continuous ($\mathcal{T}_d-$ continuous or approximately continuous) at a point x if it is continuous at x as the application from $(\mathbb{R}, \mathcal{T}_{ae})$ (from $(\mathcal{R}, \mathcal{T}_d)$) into $(\mathbb{R}, \mathcal{T}_e)$, where $\mathcal{T}_e$ denotes the

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Euclidean topology in $\mathbb{R}$. A function f is $\mathcal{T}_{ae}-$ continuous (everywhere on $\mathbb{R}$) if and only if it is $\mathcal{T}_d-$ continuous (everywhere) and almost everywhere (relative to μ) continuous [4]. A function f is said to be strongly quasi-continuous (in short s.q.c.) at a point x if for every set $A \in \mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in A \cap I$ [2].

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is s.q.c. at a point $x \in \mathbb{R}$ whenever there is an open set U such that $d_u(U, x) > 0$ and the restricted function $f/(U \cup \{x\})$ is continuous at x [3]. Now, let:

- $C = \{f; f \text{ is continuous }\}$;
- $C_{ae} = \{f; f \text{ is } \mathcal{T}_{ae}- \text{ continuous }\}$;
- $Q_s = \{f; f \text{ is s.q.c.}\}$;
- $Max_{add}(Q_s) = \{f; f + g \in Q_s \text{ for every } g \in Q_s\}$;
- $Max_{mult}(Q_s) = \{f; fg \in Q_s; \text{ for every } g \in Q_s\}$;
- $Max_{max}(Q_s) = \{f; \max(f, g) \in Q_s \text{ for every } g \in Q_s\}$;
- $Max_{min}(Q_s) = \{f; \min(f, g) \in Q_s \text{ for every } g \in Q_s\}$;
- $Max_{comp}(Q_s) = \{f; f \circ g \in Q_s \text{ for every } g \in Q_s\}$.

Remark 1 *Since all constant functions and the function $f(x) = x$ for $x \in \mathbb{R}$ belong to Q_s, we have immediately*

$$Max_{add}(Q_s) \cup Max_{mult}(Q_s) \cup Max_{max}(Q_s) \cup \\ \cup Max_{min}(Q_s) \cup Max_{comp}(Q_s) \subset Q_s.$$

Remark 2 *Since the intersection of an open set A having at a point x the density 1 and an open set B having at x positive upper density is an open set having at x positive upper density, from the elementary properties of continuous functions it follows the followig inclusions:*

- $C_{ae} \subset Max_{add}(Q_s) \cap Max_{mult}(Q_s) \cap Max_{max}(Q_s) \cap Max_{min}(Q_s)$;
- $C \subset Max_{comp}(Q_s)$.

Theorem 1 *The equality*

$$Max_{add}(Q_s) = C_{ae}$$

is true.

Proof. By Remark 2 we have the inclusion $C_{ae} \subset Max_{add}(Q_s)$. For the proof of the inclusion $Max_{add}(Q_s) \subset C_{ae}$ fix a function $f \in Max_{add}(Q_s)$. By Remark 1 the function $f \in Q_s$. If f is not in C_{ae} then there are a point $x \in \mathbb{R}$ and a positive number η such that the closure $cl(\{t; |f(t) - f(x)| > \eta\})$ of the set $\{t; |f(t) - f(x)| > \eta\}$ has positive upper density at a point x. We can assume that the closure

$$cl(\{t; f(t) > f(x) + \eta\})$$

has positive upper density at a point x. Since f belonging to Q_s is almost everywhere continuous [2, 3], we obtain

$$\mu(cl(\{t; f(t) > f(x) + \eta\}) - \{t; f(t) \geq f(x) + \eta\}) = 0$$

and consequently,

$$d_u(int(\{t; f(t) > f(x) + \eta/2\}), x) > 0.$$

Thus there is a sequence of disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; f(t) > f(x) + \eta/2\}, n = 1, 2, \ldots,$ such that:

(1) x is not in I_n for $n = 1, 2, \ldots$;

(2) f is continuous at all points $a_n, b_n, n = 1, 2, \ldots$;

(3) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;

(4) $d_u(\bigcup_n I_n, x) > 0$.

Put

$$g(t) = \begin{cases} -f(x) + \eta/2 & if \quad (t = x) \vee (t \in I_n, n = 1, 2, \ldots) \\ -f(t) & otherwise. \end{cases}$$

Then $g \in Q_s$, $f(x) + g(x) = \eta/2$, $f(t) + g(t) \geq \eta$ for $t \in I_n$, $n = 1, 2, \ldots$ and $f(t) + g(t) = 0$ otherwise on $\mathbb{R}$. So, $f + g$ is not in Q_s and consequently f is not in $Max_{add}(Q_s)$. This contradiction finishes the proof.

Theorem 2 *The equalities*

$$Max_{max}(Q_s) = Max_{min}(Q_s) = C_{ae}$$

are true.

Proof. By Remark 2 we have

$$C_{ae} \subset Max_{max}(Q_s) \cap Max_{min}(Q_s).$$

We will show only that $Max_{max}(Q_s) \subset C_{ae}$, because the proof of the inclusion $Max_{min}(Q_s) \subset C_{ae}$ is similar. Let $f \in Max_{max}(Q_s)$ be a function. By Remark 1 the function $f \in Q_s$. If f is not in C_{ae} then there are a point x and a positive number η such that

$$d_u(cl(\{t; |f(t) - f(x)| > \eta\}), x) > 0.$$

If

$$d_u(cl(\{t; f(t) > f(x) + \eta\}), x) > 0,$$

as in the proof of Theorem 1, there are disjoint closed intervals

$$I_n = [a_n, b_n] \subset \{t; f(t) > f(x) + \eta/2\},$$

such that conditions (1) – (4) from the proof of Theorem 1 are satisfied. Let

$$g(t) = \begin{cases} f(x) - \eta & if \quad (t = x) \vee (t \in I_n, n = 1, 2, \ldots,) \\ f(x) + \eta & otherwise. \end{cases}$$

Then $g \in Q_s$, $\max(f(x), g(x)) = f(x)$ and $\max(f(t), g(t)) \geq f(x) + \eta/2$ for $t \neq x$. So, $\max(f, g)$ is not in Q_s and consequently, f is not in $Max_{max}(Q_s)$. Thus

$$d_u(cl(\{t; f(t) < f(x) - \eta\}), x) > 0$$

and there are disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; f(t) < f(x) - \eta/2\}, n = 1, 2, \ldots,$ which satisfy conditions (1)–(4) from the proof of Theorem 1. Let the function g be defined the same as above. Then $g \in Q_s$, $\max(f(x), g(x)) = f(x)$, $\max(f(t), g(t)) \leq f(x) - \eta/2$ for $t \in I_n, n = 1, 2, \ldots$, and $\max(f(t), g(t)) \geq f(x) + \eta$ otherwise on $\mathbb{R}$. So, $\max(f, g)$ is not in Q_s, and consequently f is not in $Max_{max}(Q_s)$. This contradiction finishes the proof.

Theorem 3 *The equality*

$$Max_{comp}(Q_s) = C$$

is true.

Proof. By Remark 2 we have the inclusion $C \subset Max_{comp}(Q_s)$. Suppose that a function f is not continuous at a point y. Then there is a sequence of points $y_n \neq y, n = 1, 2, \ldots,$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} f(y_n) \neq f(y)$. Let $I_n = [a_n, b_n], n = 1, 2, \ldots,$ be disjoint closed intervals such that

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$;
- $a_n b_n > 0$ for $n = 1, 2, \ldots,$;
- $d_u(\bigcup_n I_n, 0) > 0$.

Put

$$g(x) = \begin{cases} y_n & if \quad x \in I_n, n = 1, 2, \ldots, \\ y & if \quad x = 0 \\ y_1 & otherwise. \end{cases}$$

Then $g \in Q_s$ and $f \circ g$ is not in Q_s, since $f \circ g$ is not s.q.c. at $x = 0$. So, $Max_{comp}(Q_s) \subset C$, and the proof is completed.

Remark 3 *If a function $f \in Q_s$ is not $\mathcal{T}_{ae}-$ continuous at a point $x \in \mathbb{R}$ at which $f(x) \neq 0$ then there is a function $g \in Q_s$ such that fg is not in Q_s.*

Proof. The same as in the proof of Theorem 1 we prove that there exist a positive real η and disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; |f(t) - f(x)| > \eta/2\}$ which satisfy conditions (1)–(4) from the proof of Theorem 1. Put

$$g(t) = \begin{cases} 1 & if \quad (t = x) \vee (t \in I_n, n = 1, 2, \ldots,) \\ 0 & otherwise. \end{cases}$$

Then $g \in Q_s$ and fg is not in Q_s, since fg is not s.q.c. at x. This completes the proof.

Remark 4 *Let $f \in Q_s$ be a function and let $x \in \mathbb{R}$ be a point such that $f(x) = 0$. If $d_u(\{t; f(t) = 0\}, x) > 0$ then for every function $g \in Q_s$ the product fg is s.q.c. at x.*

Proof. Since the functions f, g are almost everywhere continuous, the product fg is the same. Consequently, if $A = \{t; f(t)g(t) = 0$ and f, g are continuous at $t\}$ then $d_u(A, x) > 0$ and for every $\eta > 0$ we have

$$d_u(int(\{t; |f(t)g(t)| < \eta\}), x) \geq d_u(A, x) > 0.$$

So, the product fg is s.q.c. at x and the proof is completed.

To prove the following result, Remark 5, we will apply the following:

Lemma 1 *Let $A \subset \mathbb{R}$ be a closed set and let $x \in A$ be a point such that $d_u(A, x) = 0$. Then there is a sequence of disjoint closed intervals $I_n = [a_n, b_n] \subset (x - 2, x + 2)$, $n = 1, 2, \ldots$, such that:*

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;

- $d_u(\bigcup_n I_n, x) = 0$;
- $(A - \{x\}) \cap [x-1, x+1] \subset \bigcup_n int(I_n)$.

Proof. Fix a positive integer k and observe that the sets

$$B_k = [x + 1/(k+1), x + 1/k] \cap A$$

and

$$C_k = [x - 1/k, x - 1/(k+1)] \cap A$$

are compact. Let U_k, V_k be open sets such that

- $B_k \subset U_k \subset [x + (k+1)^{-1} - (4(k+1))^{-3}, x + k^{-1} + (4k)^{-3}]$;
- $C_k \subset V_k \subset [x - k^{-1} - (4k)^{-3}, x - (k+1)^{-1} + (4(k+1))^{-3}]$;
- $\mu(U_k - B_k) < \mu(B_k) + k^{-3}$;
- $\mu(V_k - C_k) < \mu(C_k) + k^{-3}$.

Since the sets B_k, C_k are compact, there are finite families of disjoint closed intervals

$$\{K_{i,k}; i = 1, \ldots, i(k)\}$$

and

$$\{L_{j,k}; j = 1, \ldots, j(k)\}$$

such that

$$B_k \subset \bigcup_{i=1}^{i(k)} int(K_{i,k}) \subset U_k$$

and

$$C_k \subset \bigcup_{j=1}^{j(k)} int(L_{j,k}) \subset V_k.$$

Then every enumeration $(I_n)_n$ of all connected components of the union

$$\bigcup_k (\bigcup_{i=1}^{i(k)} K_{i,k} \cup \bigcup_{j=1}^{j(k)} L_{j,k})$$

such that $I_i \cap I_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots$, satisfies all required conditions. So, the proof is completed.

Remark 5 *Suppose that a function $f \in Q_s$ is not $\mathcal{T}_{ae}-$ continuous at a point x at which $f(x) = 0$. If*

$$d_u(\{t; f(t) = 0\}, x) = 0$$

then there is a function $g \in Q_s$ such that the product fg is not in Q_s.

Proof. Since f is almost everywhere continuous, we obtain

$$\mu(cl(\{t; f(t) = 0\}) - \{t; f(t) = 0\}) = 0$$

and

$$d_u(cl(\{t; f(t) = 0\}), x) = 0.$$

By Lemma 1 there are disjoint closed intervals $I_n = [a_n, b_n] \subset (x-2, x+2) - \{x\}$, $n = 1, 2, \ldots$, such that

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;
- $[x-1, x+1] \cap cl(\{t; f(t) = 0\}) - \{x\} \subset \bigcup_n int(I_n)$;
- $d_u(\bigcup_n I_n, x) = 0$.

Since the function f is not $\mathcal{T}_{ae}-$ continuous at x, there are a positive real η and disjoint closed intervals $J_n = [c_n, d_n] \subset (\{t; |f(t)| \geq \eta/2\} \cap (x-1, x+1)) - \bigcup_k I_k$ such that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = x$ and $d_u(\bigcup_n J_n, x) > 0$. Moreover, we can assume that f is continuous at all points a_n, b_n, c_n, d_n, $n = 1, 2, \ldots$.

Put

$$g(t) = \begin{cases} \eta & if \quad (t = x) \vee (t \in J_n, n \geq 1) \\ 1 & if \quad (t \leq x-1) \vee (t \geq x+1) \vee (t \in I_n, n \geq 1) \\ 1/f(t) & otherwise. \end{cases}$$

It is obvious that the function g is s.q.c. at x and at every point $t \in \bigcup_n (I_n \cup J_n) \cup (-\infty, x-1] \cup [x+1, \infty)$. By elementary method we can prove that it is also s.q.c. at each point t at which $g(t) = 1/f(t)$. So, $g \in Q_s$. But the product fg is not s.q.c. at x, since $f(x)g(x) = 0$, $f(t)g(t) = 1$ for $t \in (x-2, x+2) - \bigcup_n (I_n \cup J_n) - \{x\}$, $|f(t)g(t)| \geq \eta^2/2$ for $t \in J_n$, $n \geq 1$, and $d_u(\bigcup_n I_n, x) = 0$. This finishes the proof.

From Remarks 1 – 5 it follows immediately:

Theorem 4 *A function $f \in Max_{mult}(Q_s)$ if and only if it is in Q_s and satisfies the following condition:*

(F) if f is not $\mathcal{T}_{ae}-$ continuous at a point x then $f(x) = 0$ and $d_u(\{t; f(t) = 0\}, x) > 0$.

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