Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 622–630

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THE HAKE'S PROPERTY FOR SOME INTEGRALS OVER MULTIDIMENSIONAL INTERVALS

1 Introduction

The Hake's property is named after the work of H. Hake [4] in 1921, which was an important step in proving the equivalence between the integrals of Denjoy and Perron (see e.g. Chapter VIII in [13]). For a real function f defined on a compact interval $[a,b] \subseteq \mathbb{R}$, H. Hake proved the equivalence between the Perron-integrability of f over $[a,b]$ and its Perron-integrability over $[a,c]$ for all $a < c < b$ together with the existence of the limit

$$\lim_{c \nearrow b} \int_a^c f.$$

This shows in particular that improper integrals are proper ones in the frame of Perron integration.

The Riemann-type definition of the Perron integral introduced independently by J. Kurzweil [6] and R. Henstock [5] has led to various extensions in the multidimensional case. For these integrals, the formulation and proof of a Hake's property is an interesting question and results in this direction have been recently obtained by J. Mawhin and W. F. Pfeffer for the Pfeffer integral [10], and by J. Kurzweil and J. Jarnik for the α-regular integral [8].

For a real function f defined on a compact interval $I \subseteq \mathbb{R}^m$, we prove in this note the equivalence between the integrability of f over I and the integrability of f over all intervals $J \subseteq I^\circ$ together with the existence of the (appropriate) limit

$$\lim_{F \nearrow I^\circ} \int_F f,$$

Key Words: Hake's Property, Kurzweil-Henstock Integral
Mathematical Reviews subject classification: Primary: 26B15 Secondary: 26A39
Received by the editors June 30, 1994
*Supported by a grant from the Swiss National Funds for Scientific Research.

where $F \subseteq I^\circ$ is a figure (i.e. a finite union of compact intervals of $\mathbb{R}^m$). More generally, instead of the interior I° of I, we consider an arbitrary open subset of I.

The definition of our integral covers the cases of the Kurzweil-Henstock integral (KH-integral), of the integrals of J. Mawhin (GP-integral) and W. F. Pfeffer (Pf-integral), and of the α-regular integral. So the Hake's property expressed by Theorem 3.3 contains as particular cases the two recent results mentioned before. For the one-dimensional integral, it also encompasses the Harnack's property.

As shown in Section 4, the Hake's property proved in this note can be used to obtain counterexamples in a very simple way. We give three examples : a function which is α_1-regularly integrable for all $\alpha \leq \alpha_1 < 1$ but not α_2-regularly integrable for all $0 < \alpha_2 < \alpha$, a GP-integrable function which is not Pf-integrable, and a Pf-integrable function which is not KH-integrable (and even not M_1-integrable, cf. [2]).

2 Preliminaries

Throughout this paper $I = [a_1, b_1] \times \ldots \times [a_m, b_m]$ is a non-degenerate compact interval of $\mathbb{R}^m$. On I we consider the metric $d(x,y) = \max |y_i - x_i|$. The interior of a subset $E \subseteq I$ is denoted by $int E$ and its boundary by $bd\, E$. We use the notations I° and ∂I for the interior and the boundary of the interval I with respect to the topology of $\mathbb{R}^m$.

The set of all (non-degenerate) compact subintervals $J \subseteq I$ is denoted by $\mathcal{J}(I)$. Given an interval $J = [c_1, d_1] \times \ldots \times [c_m, d_m]$ its *measure*, *length* and *thickness* are the respective positive numbers

$$m(J) = \prod_{i=1}^m (d_i - c_i),\ \ell(J) = \max(d_i - c_i) \text{ and } t(J) = \min(d_i - c_i).$$

The *regularity* of J is the number $r(J) = t(J) \cdot \ell(J)^{-1}$. More generally, we shall consider four regularity functions $\rho : \mathcal{J}(I) \to [0,1]$, which correspond to the four different integrals mentioned in the introduction (cf. Definition 2.2 below).

A subset $F \subseteq I$ is a *figure* if it is a finite union of elements of $\mathcal{J}(I)$. A *partition* of a figure F is a finite collection of non-overlapping intervals $J_1, \ldots, J_r \in \mathcal{J}(I)$ with $F = \bigcup_{i=1}^r J_i$.

A *system* on a subset $E \subseteq I$ is a finite collection $\{(J_1, x_1), \ldots, (J_r, x_r)\}$, usually noted S, where $J_1, \ldots, J_r \in \mathcal{J}(I)$ are non-overlapping intervals and $x_i \in J_i \cap E$ for all $i = 1, \ldots, r$. A system is called a *division* of the interval I if one has $I = \bigcup_{i=1}^r J_i$. And we say that a system S *completes* a figure $F \subseteq I$

if the intervals of S do not overlap the figure F and $I = \left(\bigcup_{i=1}^{r} J_i\right) \cup F$.

A *gauge* on a subset $E \subseteq I$ is any positive map $\delta : E \to \mathbb{R}_+$. Given a parameter of regularity $0 < \alpha < 1$ and a gauge $\delta : E \to \mathbb{R}_+$ one says that a system S on E is *(α, δ)-fine* if it satisfies the conditions

$$\rho(J_i) \geq \alpha \text{ and } \ell(J_i) \leq \delta(x_i) \text{ for all } i = 1, \ldots, r.$$

The set of (α, δ)-fine divisions of I is denoted by $\mathcal{D}(I, \alpha, \delta)$. The following lemma ensures that the definition of the integral is meaningful:

Lemma 2.1. (Cousin) *For every choice of α and δ there exists a division D of I which is (α, δ)-fine, i.e. the set $\mathcal{D}(I, \alpha, \delta)$ is not empty.*

Definition 2.2. One says that a function $f : I \to \mathbb{R}$ is *integrable* if there exists a number $A \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $0 < \alpha < 1$ there exists a gauge $\delta : I \to \mathbb{R}_+$ with the property

$$|S(f, D, I) - A| < \varepsilon \text{ for every division } D \in \mathcal{D}(I, \alpha, \delta),$$

where $S(f, D, I) = \sum_{i=1}^{r} f(x_i) m(J_i)$ is the usual Riemann sum associated with D. The *integral* $A \in \mathbb{R}$ is unique and denoted by $\int_I f$.

We consider four particular choices for the regularity function ρ:

1) $\rho(J) = r(J)$. One gets the GP-integral, cf. Definition 9 in [9].

2) $\rho(J) = \inf\{d_i - c_i \,/\, a_i \neq c_i \text{ and } d_i \neq b_i\} \cdot \ell(J)^{-1}$, and $\rho(J) = 1$ if one corner of I is contained in J. According to a result of J. Kurzweil and J. Jarnik [7] one gets the Pf-integral of W. F. Pfeffer [12].

3) $\rho(J) = 1$. Then one can leave the parameter of regularity α and one gets the classical Kurzweil-Henstock integral.

4) $\rho(J) = r(J)$ if $r(J) \geq \alpha_0$ and $\rho(J) = 0$ otherwise. Then the parameter α can be taken as a constant and one gets the α_0-regular integral [8].

We shall use the two following propositions (which are well-known for these integrals).

Remark 2.3. By considering the map ρ as a function of tagged intervals (J, x) instead of a function of intervals J, one may also include other integrals, like for instance the integral of Z. Buczolich in the plane [1].

Proposition 2.4. *Let $f : I \to \mathbb{R}$ be an integrable function. Then f is integrable on every subinterval $J \in \mathcal{J}(I)$ and the indefinite integral $\varphi(J) = \int_J f$ is an additive function of intervals.*

Proposition 2.5. (Saks-Henstock) *Let $f : I \to \mathbb{R}$ be an integrable function and let $\delta : I \to \mathbb{R}_+$ be a gauge such that $|S(f, D, I) - \int_I f| \leq \varepsilon$ for every (α, δ)-fine division D of the interval I. Then one has*

$$\left|\sum_{i=1}^{r}\left\{f(x_i)m(J_i)-\int_{J_i}f\right\}\right|\le\varepsilon.$$

for every (α,δ)-fine system $S=\{(J_1,x_1),\ldots,(J_r,x_r)\}$ on the interval I.

3 The main theorem

Let $E\subseteq I$ be a closed subset such that $\emptyset\ne E\ne I$, and suppose that $f:I\to\mathbb{R}$ is integrable on every interval $J\in\mathcal{J}(I)$ contained in $I\setminus E$. Given any figure $F\subseteq I\setminus E$ one can define the integral of f on the figure F by

$$\int_F f=\sum_{i=1}^{r}\int_{J_i}f,\text{ where } J_1,\ldots,J_r \text{ is any partition of } F.$$

Since the integral is (weakly) additive, cf. Proposition 2.4, the value of $\int_F f$ does not depend on the choice of the partition.

Definition 3.1. Given a parameter $0<\alpha<1$ and a gauge $\delta:E\to\mathbb{R}_+$ we say that a figure $F\subseteq I\setminus E$ is (α,δ)-*close* to the subset $I\setminus E$ if there exists a system S on E which is (α,δ)-fine and completes F. We denote by $\mathcal{F}(I\setminus E,\alpha,\delta)$ the set of all figures $F\subseteq I\setminus E$ which are (α,δ)-close to $I\setminus E$.

Defining $\delta(x)=\frac{1}{2}d(x,E)$ for all $x\in I\setminus E$ and using the Cousin's lemma one easily shows that the set $\mathcal{F}(I\setminus E,\alpha,\delta)$ is not empty for every choice of α and δ.

Definition 3.2. We say that the integral $\int_F f$ *converges* to the number $A\in\mathbb{R}$ when the figure F tends to $I\setminus E$ if for any $\varepsilon>0$ and any parameter $0<\alpha<1$ there exists a gauge $\delta:E\to\mathbb{R}_+$ with the property

$$\left|\int_F f-A\right|<\varepsilon \text{ for every figure } F\in\mathcal{F}(I\setminus E,\alpha,\delta).$$

Theorem 3.3. *Let $E\subseteq I$ be a closed subset and suppose that $f:I\to\mathbb{R}$ is integrable on every interval $J\subseteq I\setminus E$. If the function $f\cdot\chi_E$ is integrable on I and if the integral $\int_F f$ converges to a limit $A\in\mathbb{R}$ when the figure F tends to $I\setminus E$, then f is integrable on I and $\int_I f=\int_I f\cdot\chi_E+A$.*

Proof. By considering $f-f\cdot\chi_E$ one may assume that $f(x)=0$ for all $x\in E$. We want to show that A is the integral of f. We first choose a sequence of intervals $I_n\in\mathcal{J}(I)$ such that $I_n\subseteq I\setminus E$ for each $n\in\mathbb{N}$ and $I\setminus E=\bigcup_{n=1}^{\infty}\mathit{int}\,I_n$. For instance, one can take the family of all closed balls $B\big(x,\frac{1}{2}d(x,E)\big)$, where x runs over the rational points of $I\setminus E$.

Given $\varepsilon>0$ and $0<\alpha<1$ there exists for each $n\in\mathbb{N}$ a gauge $\delta_n:I_n\to\mathbb{R}_+$ with

$$\left|S(f,D,I_n)-\int_{I_n}f\right|<2^{-n}\varepsilon \text{ for every division } D\in\mathcal{D}(I_n,\alpha,\delta_n).$$

One may assume that $\delta_n(x)<d(x,E)$ for every $x\in I_n$ and $\delta_n(x)\le d(x,\mathit{bd}\,I_n)$ for every $x\in\mathit{int}\,I_n$. For $x\in E_n:=\mathit{int}\,I_n\setminus\bigcup_{j=1}^{n-1}\mathit{int}\,I_j$ we define $\delta(x)=\delta_n(x)$.

By hypothesis there exists a gauge $\delta : E \to \mathbb{R}_+$ such that $|\int_F f - A| < \varepsilon$ for every figure $F \in \mathcal{F}(I \setminus E, \alpha, \delta)$. One thus gets a gauge $\delta : I \to \mathbb{R}_+$.

Now let D be any (α, δ)-fine division of the interval I. We consider the figure $F = \bigcup\{J_i \,/\, x_i \notin E\}$. By construction one has $F \in \mathcal{F}(I \setminus E, \alpha, \delta)$. Then

$$|S(f, D, I) - A| \leq \left|\sum_{x_i \notin E} \left\{f(x_i)m(J_i) - \int_{J_i} f\right\}\right| + |\int_F f - A| \leq$$

$$\sum_{n=1}^{\infty} \left|\sum_{x_i \in E_n} \left\{f(x_i)m(J_i) - \int_{J_i} f\right\}\right| + |\int_F f - A| < \varepsilon + \varepsilon = 2\varepsilon$$

according to Proposition 2.5 (recall that $J_i \subseteq I_n$ for every i with $x_i \in E_n$), and this proves that the function f is integrable on I. □

Remark 3.4. The theorem contains as a particular case a result of J. Kurzweil and J. Jarnik for the α-regular integral, cf. Proposition 2 in [8]. And for $m = 1$ it also involves the well-known Harnack's property (left as exercise).

Example 3.5. Let $E = \partial I$ be the boundary of I for the topology of $\mathbb{R}^m$. Since E is of Lebesgue-measure zero one has $\int_I f \cdot \chi_E = 0$ for any function $f : I \to \mathbb{R}$. So the theorem reads as follows: If the function f is integrable on every interval $J \subseteq I^\circ$ and the integral $\int_F f$ converges to a limit $A \in \mathbb{R}$ when the figure F tends to I°, then f is integrable on the interval I and $\int_I f = A$.

In that situation a sufficient condition for the Pf-integrability of f was given in [10]. One can prove that if $\int_F f$ converges to a limit $A \in \mathbb{R}$ in the sense of [10], then it converges to A in our sense. But this is an immediate consequence of the following lemma, which says that the condition of Theorem 3.3 is also necessary.

Lemma 3.6. *Let $E \subseteq I$ be a closed subset and let $f : I \to \mathbb{R}$ be such that both f and $f \cdot \chi_E$ are integrable on the interval I. Then the integral $\int_F f$ converges to the number $\int_I (f - f \cdot \chi_E)$ when the figure F tends to $I \setminus E$.*

Proof. One may assume that $f(x) = 0$ for all $x \in E$. Given $\varepsilon > 0$ and $0 < \alpha < 1$ there exists a gauge $\delta : I \to \mathbb{R}_+$ with the property

$$|S(f, D, I) - \int_I f| < \varepsilon \text{ for every division } D \in \mathcal{D}(I, \alpha, \delta).$$

Now consider any figure $F \in \mathcal{F}(I \setminus E, \alpha, \delta)$. We choose a partition $J_1, \dots, J_r$ of F and for each $i = 1, \dots, r$ a division $D_i \in \mathcal{D}(J_i, \alpha, \delta)$. By definition there exists a system S on E which is (α, δ)-fine and completes F. Then $D = \left(\bigcup_{i=1}^r D_i\right) \cup S$ is a division of the interval I and $D \in \mathcal{D}(I, \alpha, \delta)$. So we obtain

$$|\int_F f - \int_I f| \leq \left|\sum_{i=1}^r \left\{\int_{J_i} f - S(f, D_i, J_i)\right\}\right| + |S(f, D, I) - \int_I f| < 2\varepsilon$$

according to Proposition 2.5 (and the additivity of the integral, cf. Proposition 2.4), and thus the lemma is proved. □

4 Some examples

Example 4.1. Let $I = [0,1] \times [0,1]$ be the unit square. Given some parameter $0 < \alpha < 1$ we consider the function $f : I \to \mathbb{R}$ defined by

$$f(x,y) = 1/x^3 \text{ if } 0 < y < \tfrac{1}{2}\alpha x,\; f(x,y) = -1/x^3 \text{ if } \tfrac{1}{2}\alpha x < y < \alpha x$$

and $f(x,y) = 0$ otherwise. Then f is α_1-regularly integrable for any $\alpha \leq \alpha_1 < 1$ but not α_2-regularly integrable for any $0 < \alpha_2 < \alpha$.

Proof. We take $E = \{(0,0)\}$. Therefore we are interested in figures $F = I \setminus \mathit{int}\, J$ with $J = [0,x] \times [0,y]$. Clearly, one has $\int_F f = 0$ for every figure $F \in \mathcal{F}(I \setminus E, \alpha_1, 1)$, and this proves that f is α_1-regularly integrable.

Now let $\frac{1}{2}\alpha \leq \alpha_2 < \alpha$ and consider the interval $J = [0,\delta] \times [0,\alpha_2\delta]$. An easy calculation shows that $\int_F f = -(\alpha - \alpha_2)^2/2\alpha_2\delta$. Therefore $\int_F f$ cannot converge when the figure F tends to $I \setminus E$, and this proves that f is not α_2-regularly integrable. In particular, f is not α_2-regularly integrable for $0 < \alpha_2 < \frac{1}{2}\alpha$. □

Example 4.2. Let $I = [0,1] \times [0,1]$ be the unit square. We consider the function $f : I \to \mathbb{R}$ defined by

$$f(x,y) = 1/x^4 \text{ if } 0 < y < \tfrac{1}{2}x^2,\; f(x,y) = -1/x^4 \text{ if } \tfrac{1}{2}x^2 < y < x^2$$

and $f(x,y) = 0$ otherwise. Then f is GP-integrable but not Pf-integrable.

Proof. As before, we take $E = \{(0,0)\}$. We first show that f is GP-integrable. For any figure $F = I \setminus \mathit{int}\, J$ with $F \in \mathcal{F}(I \setminus E, \alpha, \alpha)$ one has $y \geq \alpha x \geq x^2$ because $r(J) \geq \alpha$ and $\ell(J) \leq \alpha$. Therefore $\int_F f = 0$ and the assertion follows.

But for the Pf-integral one has $\rho(J) = 1$ for *every* interval $J = [0,x] \times [0,y]$. In particular, for $J = [0,\delta] \times [0, \frac{1}{2}\delta^2]$ we calculate that $\int_F f = -(4\sqrt{2} - 5)/6\delta$, and this proves that f is not Pf-integrable. □

Example 4.3. Let $I = [0,1] \times [0,1]$ be the unit square. We consider the function $f : I \to \mathbb{R}$ defined by

$$f(x,y) = (-1)^i\, 2^n \text{ if } (x,y) \in \left(\frac{i-1}{4^n}, \frac{i}{4^n}\right] \times \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \text{ (for } i = 1, \ldots, 4^n)$$

and $f(x,y) = 0$ if $xy = 0$. Then f is Pf- integrable but not KH-integrable.

Proof. By Fubini's theorem the function f is not Kurzweil-Henstock integrable because the partial integral $\int_0^1 f(x,y)\,dy$ does not exist for *any* $x \in (0,1]$. In order to show that f is Pf-integrable we apply Theorem 3.3 with $E = [0,1] \times \{0\}$. We shall use the intervals $I_n = [0,1] \times [2^{-n}, 2^{-n+1}]$.

Let $\varepsilon > 0$ and $0 < \alpha < 1$ be given. We choose a constant gauge $\delta = 2^{-N+1}$ on E (the integer N will be precised later). Obviously, any figure

$F \in \mathcal{F}(I \setminus E, \alpha, \delta)$ contains the interval $[0,1] \times [2^{-N+1}, 1]$, and therefore

$$\int_F f = \sum_{n=N}^{m} \int_{F \cap I_n} f,$$

where $I_1, \dots, I_m$ are the intervals with $F \cap int\, I_n \neq \emptyset$. Let S be a system on E which is (α, δ)-fine and completes F. Say $(0,0) \in J_1$ and $(1,0) \in J_r$. Using that $\int_{I_n} f = 0$ for every $n \in \mathbb{N}$ we obtain the following inequality:

$$|\int_{F \cap I_n} f| \leq \sum_{i=1}^{r} |\int_{J_i \cap I_n} f|.$$

If $J_i \cap int\, I_n \neq \emptyset$, then $\ell(J_i) > 2^{-n}$, and for $i = 2, \dots, r-1$ this implies $t(J_i) > \alpha \cdot 2^{-n}$. So we deduce that there exist at most $[2^n/\alpha] + 2$ intervals J_i with $J_i \cap int\, I_n \neq \emptyset$. And since $|\int_{J_i \cap I_n} f| \leq 4^{-n}$ we conclude that

$$|\int_{F \cap I_n} f| \leq 1/\alpha 2^n + 2/4^n.$$

Thus it suffices to choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} (1/\alpha 2^n + 2/4^n) < \varepsilon$. Hence $\int_I f = 0$. □

In fact, this function is an example of a Pf-integrable function which is not M_1-integrable (another such example was given in [2]). Before we prove this result, we first recall the definition of the M_1-integral for the two-dimensional case (compare with [3]).

Definition 4.4. A function $f : I \to \mathbb{R}$ is called *M_1-integrable* if there exists a number $A \in \mathbb{R}$ such that for any $\varepsilon > 0$ and any $K > 0$ there exists a gauge $\delta : I \to \mathbb{R}_+$ with the property $|S(f, D, I) - A| < \varepsilon$ for every division D of I satisfying

$$\ell(J_i) \leq \delta(x_i) \text{ for all } i = 1, \dots, r \text{ and } \sum_{i=1}^{r} \ell(J_i)^2 \leq K,$$

noted $D \in \mathcal{D}_1(I, K, \delta)$. Clearly, any KH-integrable function is M_1-integrable. And in [11] D. J. F. Nonnenmacher proved that any M_1-integrable is Pf-integrable.

Proposition 4.5. *The function $f : I \to \mathbb{R}$ of Example 4.3 is not M_1-integrable.*

Proof. We show that for any gauge $\delta : I \to \mathbb{R}_+$ there exist two divisions D_1 and D_2 in $\mathcal{D}_1(I, 5, \delta)$ with $|S(f, D_1, I) - S(f, D_2, I)| \geq \frac{1}{3}$. We first cover the set $E = [0,1] \times \{0\}$.

Let $\gamma(x) := \frac{1}{4}\delta(x, 0)^2$. There exists a division $\{(A_1, x_1), \dots, (A_s, x_s)\}$ of the interval $[0,1] \subseteq \mathbb{R}$ with the property

$$A_i = \left[\frac{m_i - 1}{4^{n_i}}, \frac{m_i}{4^{n_i}}\right] \text{ and } \ell(A_i) = \frac{1}{4^{n_i}} \leq \gamma(x_i) \text{ for all } i = 1, \dots, s.$$

We put $N_1 = \{i \,/\, m_i \text{ is odd}\}$ and $N_2 = \{i \,/\, m_i \text{ is even}\}$. Then we have $\sum_{i \in N_1} \ell(A_i) \geq \frac{1}{2}$ or $\sum_{i \in N_2} \ell(A_i) \geq \frac{1}{2}$. Say $\sum_{i \in N_1} \ell(A_i) \geq \frac{1}{2}$.

As $4^{-n_i} \le \gamma(x_i)$ we obtain $2^{-n_i+1} \le \delta(x_i, 0)$. Considering the intervals $B_i = [0, 2^{-n_i}]$ and $C_i = [2^{-n_i}, 2^{-n_i+1}]$, we construct the division D_1 as follows:

- a system $\{(J_1, \xi_1), \dots, (J_s, \xi_s)\}$ on the set E, where $J_i = A_i \times (B_i \cup C_i)$ if $i \in N_1$ and $J_i = A_i \times B_i$ if $i \in N_2$, and $\xi_i = (x_i, 0)$;
- a system $\{(J_{s+1}, \xi_{s+1}), \dots, (J_r, \xi_r)\}$ which completes the preceding one, and such that $r(J_i) = 1$ and $\ell(J_i) \le \delta(\xi_i)$ for all $i = s+1, \dots r$.

Clearly, one has $\ell(J_i) \le \delta(\xi_i)$ for all $i = 1, \dots, r$. And since $\sum_{i=1}^r \ell(J_i)^2 \le \sum_{i=1}^s 4^{-n_i+1} + \sum_{i=s+1}^r m(J_i) \le \sum_{i=1}^s 4\,\ell(A_i) + \sum_{i=1}^r m(J_i) = 5$ one gets $D_1 \in \mathcal{D}_1(I, 5, \delta)$.

For the second division $D_2 \in \mathcal{D}_1(I, 5, \delta)$ we put the intervals $A_i \times B_i$ in place of the intervals $J_i = A_i \times (B_i \cup C_i)$, and we add (for each $i \in N_1$) a division E_i of the interval $A_i \times C_i$, which satisfies the following properties:

1. $r(J_{ij}) = 1$ and $\ell(J_{ij}) \le \delta(\xi_{ij})$ for all j, and
2. $|S(f, E_i, A_i \times C_i) - \int_{A_i \times C_i} f| \le \frac{1}{6s}$.

Finally, since $\int_{A_i \times C_i} f = -4^{-n_i} = -\ell(A_i)$, we conclude that

$$|S(f, D_1, I) - S(f, D_2, I)| = |\sum_{i \in N_1} S(f, E_i, A_i \times C_i)| \tag{1}$$

$$\ge \sum_{i \in N_1} \ell(A_i) - \sum_{i \in N_1} \frac{1}{6s}, \tag{2}$$

and this proves that $|S(f, D_1, I) - S(f, D_2, I)| \ge \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. Therefore the function f cannot be M_1-integrable on the interval I. □

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