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HAUSDORFF MEASURE ON PERTURBED CANTOR SETS

1 Introduction

We recall the definition of a perturbed Cantor set from [1]. Let $I_\phi = [0,1]$. We obtain the left subinterval $I_{\sigma,1}$ and the right subinterval $I_{\sigma,2}$ of I_σ by deleting a middle open subinterval of I_σ inductively for each $\sigma \in \{1,2\}^n$, where $n = 0,1,2,\cdots$. Consider $E_n = \cup_{\sigma\in\{1,2\}^n} I_\sigma$. Then $\{E_n\}$ is a decreasing sequence of sets. For each n we set $|I_{\sigma,1}|/|I_\sigma| = a_{n+1}$ and $|I_{\sigma,2}|/|I_\sigma| = b_{n+1}$ for all $\sigma \in \{1,2\}^n$, where $|I|$ denotes the diameter of I. We call $F = \bigcap_{n=0}^{\infty} E_n$ a perturbed Cantor set.

We assume the sequences of ratios $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$, where $d_n = 1-(a_n+b_n)$, are uniformly bounded away from 0. In [1] it was shown how to find the Hausdorff dimension of a perturbed Cantor set. Here we investigate the value of the s-dimensional Hausdorff measure of a perturbed Cantor set. We recall the s-dimensional Hausdorff measure of F, $H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$, where $H^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} |U_n|^s : \{U_n\}_{n=1}^{\infty}$ is a δ-cover of $F\}$, and the Hausdorff dimension of F, $\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}(= \inf\{s > 0 : H^s(F) = 0\})$. (See [2].) We note if $\{a_n\}$ and $\{b_n\}$ are given, then a perturbed Cantor set F is determined and *vice versa*[1]. We recall the set function $h^s(F) = \liminf_{n\to\infty} \prod_{k=1}^{n}(a_k^s + b_k^s)(= \liminf_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s)$ for $s \in (0,1)$ and a perturbed Cantor set F. There is a close connection between the set functions h^s and H^s. Hence in [1] we investigated the Hausdorff dimension of the aforementioned perturbed Cantor set using h^s. Even though we have information on the Hausdorff dimension, s, of a perturbed Cantor set, we are curious about the value of the corresponding s-dimensional Hausdorff measure of the set. Further we wonder what form of subset of the perturbed Cantor set has positive and finite s-dimensional Hausdorff measure. In Theorem 1 we give concrete examples of such subsets obtained by eliminating the right

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subintervals in some steps of the constructing the perturbed Cantor set. Using these subsets, we can find that h^s is equivalent to H^s. For this, one needs to observe that $H^s(F) \leq h^s(F)$, and it suffices to show that if $h^s(F) = \infty$, then $H^s(F) = \infty$ and if $0 < h^s(F) < \infty$, then $0 < H^s(F) < \infty$. In fact we find that the set function h^s is intimately related to H^s. Finally we give applications of our result.

2 Main Results

In this section F denotes a perturbed Cantor set. Now, constructing a proper mass distribution μ on a subset E of F, we use the Hausdorff density theorem with μ to give a lower bound on the Hausdorff measure of E.

Theorem 1 *Let $h^s(F) = \infty$. Given any number $\beta > 0$, there is a subset E of F such that $\beta < H^s(E) < \infty$.*

PROOF. Since $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ are uniformly bounded away from 0, we may assume that $a_n, b_n, d_n > \alpha$ for some $\alpha > 0$ for all n.

Let $n_0 = 0$, and $n_i = \max\{n > n_{i-1} : \prod_{j=1}^{n}(a_j^s+b_j^s) = \inf_{m>n_{i-1}} \prod_{j=1}^{m}(a_j^s+b_j^s)\}$ for $i = 1, 2, 3, \cdots$ inductively. Since $h^s(F) = \infty$, the set on which n_i is defined is nonempty and finite so n_i is well-defined. Let $y_i = \prod_{j=1}^{n_i}(a_j^s + b_j^s)$ for $i = 1, 2, 3, \cdots$. Fix p such that $y_p \left(\frac{\alpha^{4s}}{4}\right) > \beta$ and let $i_1 = p$ and $x_{0,n} = \prod_{j=1}^{n}(a_j^s + b_j^s)$. Set

$$x_{k,n} = \begin{cases} x_{k-1,n} & \text{if } n \leq n_{i_k} \\ \prod_{j=1}^{n}(a_j^s + z_k(j)b_j^s) & \text{if } n \geq n_{i_k} + 1 \end{cases}$$

with

$$z_k(j) = \begin{cases} 0 & \text{if } j = n_{i_l} + 1, \text{ where } l = 1, 2, \cdots, k \\ 1 & \text{otherwise} \end{cases}$$

and $i_{k+1} = \max\{i > i_k : x_{k,n_i} \leq y_p\}$ for $k = 1, 2, 3, \cdots$ inductively. $1 \leq \sharp\{i > i_k : x_{k,n_i} \leq y_p\} < \infty$ follows from

$$x_{k,n_{i_k+1}} \leq x_{k,n_{i_k}+1} = x_{k,n_{i_k}} a^s_{n_{i_k}+1} = x_{k-1,n_{i_k}} a^s_{n_{i_k}+1} < y_p$$

and $\lim_{n\to\infty} \inf_{m>n} x_{k,m} = \infty$. We define $z : \mathbb{N} \to \{0, 1\}$ by

$$z(j) = \begin{cases} 0 & \text{if } j = n_{i_l} + 1, \text{ where } l = 1, 2, \cdots \\ 1 & \text{otherwise.} \end{cases}$$

If $m \geq n_p + 1$, then there exists k such that $n_{i_k} + 1 \leq m \leq n_{i_{k+1}}$ and $\prod_{j=1}^{m}(a_j^s + z(j)b_j^s) = x_{k,m}$.

Since $x_{k,n_{i_k}+1} \leq x_{k,n_{i_k}+j}$ for all $j = 1, 2, \cdots$, we get $x_{k,m} \geq x_{k,n_{i_k}+1}$. Thus

$$x_{k,m} \geq x_{k,n_{i_k}+1} = x_{k-1,n_{i_k}+1} \frac{a^s_{n_{i_k}+1}}{a^s_{n_{i_k}+1} + b^s_{n_{i_k}+1}} > y_p \frac{\alpha^s}{2}.$$

Therefore, $\prod_{j=1}^{m}(a_j^s + z(j)b_j^s) > y_p \frac{\alpha^s}{2}$ for all $m = n_p + 1, n_p + 2, n_p + 3, \cdots$.

Moreover, $\prod_{j=1}^{n_{i_k}+1}(a_j^s + z(j)b_j^s) = x_{k,n_{i_k}+1} < y_p$ for $k = 1, 2, \cdots$ since $n_{i_k} + 1 \leq n_{i_k+1} \leq n_{i_{k+1}}$ and since $\prod_{j=1}^{n_{i_k}+1}(a_j^s + z(j)b_j^s) = \prod_{j=1}^{n_{i_k}+1}(a_j^s + z_k(j)b_j^s)$. Hence $\prod_{j=1}^{n}(a_j^s + z(j)b_j^s) < y_p$ for infinitely many n. Therefore $y_p \dfrac{\alpha^s}{2} \leq \liminf_{n\to\infty} \prod_{j=1}^{n}(a_j^s + z(j)b_j^s) \leq y_p$.

Now we find a sequence $\{z_k\}_{k=1}^{\infty}$ such that

$$z_1 = \min z^{-1}(0), z_{k+1} = \min[z^{-1}(0) \setminus \{z_1, \cdots, z_k\}]$$

for $k = 1, 2, \cdots$. Thus we can define

$$I^*_{i_1,\cdots,i_k} = \begin{cases} \phi & \text{if } i_{z_j} = 2 \text{ for some } j \text{ such that } 1 \leq j \leq \max\{l : z_l \leq k\} \\ I_{i_1,\cdots,i_k} & \text{otherwise} \end{cases}$$

for each $k = 1, 2, \cdots$. Let $E_n^* = \cup\{I_\sigma^* : \sigma \in \{1,2\}^n\}$. Put $\mu(I_\sigma) = \sum_{I \in I_\sigma \cap E_k^*} |I|^s$ for each $\sigma \in \{1,2\}^k$, where $k = 1, 2, \cdots$. Note that

$$\mu(I_\sigma) = |I_\sigma^*|^s \liminf_{n\to\infty} \prod_{i=k+1}^{n} (a_i^s + z(i)b_i^s).$$

Then μ extends to a mass distribution on $[0,1]$ whose support is in E $= \bigcap_{n=1}^{\infty} E_n^* \subset F$ (cf. Proposition 1.7 [2]) since

$$\begin{aligned} \mu(I_\sigma) &= |I_\sigma^*|^s (a_{k+1}^s + z(k+1)b_{k+1}^s) \liminf_{n\to\infty} \prod_{i=k+2}^{n} (a_i^s + z(i)b_i^s) \\ &= (|I_{\sigma,1}^*|^s + |I_{\sigma,2}^*|^s) \liminf_{n\to\infty} \prod_{i=k+2}^{n} (a_i^s + z(i)b_i^s) \\ &= \mu(I_{\sigma,1}) + \mu(I_{\sigma,2}). \end{aligned}$$

Clearly

$$y_p \frac{\alpha^s}{2} \leq \mu([0,1]) = \liminf_{n\to\infty} \prod_{k=1}^{n} (a_k^s + z(k)b_k^s) \leq y_p.$$

Let $x \in F = \bigcap_{n=1}^{\infty} E_n$. Then there is a sequence $\{I_{\sigma_n}\}_{n=1}^{\infty}$, where $\sigma_n \in \{1,2\}^n$ such that $\bigcap_{n=1}^{\infty} I_{\sigma_n} = \{x\}$. Given a small positive number r, there exists k such that $|I_{\sigma_{k+1}}| \le r < |I_{\sigma_k}|$. Since $d_{j+1}|I_{\sigma_j}| \ge \alpha |I_{\sigma_k}| > \alpha r$ for each j such that $0 \le j \le k$, $B_{\alpha r}(x) \subset [\bigcup_{\tau(\neq \sigma_k) \in \{1,2\}^k} I_\tau]^c$, where $B_{\alpha r}(x)$ is the ball of radius αr with center x. Thus $\mu(B_{\alpha r}(x)) \le \mu(I_{\sigma_k})$. Now,

$$\frac{\mu(B_{\alpha r}(x))}{(\alpha r)^s} \le \frac{\mu(I_{\sigma_k})}{\alpha^s |I_{\sigma_{k+1}}|^s} \le \frac{\mu(I_{\sigma_k})}{\alpha^s(\alpha^s |I_{\sigma_k}|^s)}$$
$$\le \frac{|I_{\sigma_k}|^{s-s}}{\alpha^{2s}} \liminf_{n\to\infty} \prod_{i=k+1}^{n} (a_i^s + z(i) b_i^s).$$

Since $y_p \frac{\alpha^s}{2} \le \liminf_{n\to\infty} \prod_{k=1}^{n} (a_k^s + z(k) b_k^s) \le y_p$, the sequence

$$\{\liminf_{n\to\infty} \prod_{i=k+1}^{n} (a_i^s + z(i) b_i^s)\}_{k=n_p}^{\infty}$$

has an upper bound $\dfrac{y_p}{y_p \frac{\alpha^s}{2}} = \dfrac{2}{\alpha^s}$. Then $\limsup_{r\to 0} \dfrac{\mu(B_r(x))}{r^t} \le \dfrac{\left(\frac{2}{\alpha^s}\right)}{\alpha^{2s}} = \dfrac{2}{\alpha^{3s}}$.

Thus $H^s(E) \ge \dfrac{\alpha^{3s}}{2} \mu(E) \ge \dfrac{\alpha^{3s}}{2}(y_p \dfrac{\alpha^s}{2}) = y_p \dfrac{\alpha^{4s}}{4} > \beta$ by Proposition 4.9 [2]. Clearly $H^s(E) \le y_p$. □

From Theorem 1, we immediately have the following.

Corollary 2 *If $h^s(F) = \infty$, then $H^s(F) = \infty$.*

Remark 1 *We could also prove Corollary 2 using a natural mass distribution on F. (See [1].) Combining this fact and exercise 4.8 of [2], we also have Theorem 1 although we don't have a constructive proof.*

Theorem 3 *If $0 < h^s(F) < \infty$, then $0 < H^s(F) < \infty$.*

PROOF. Since $h^s(F) > 0$, there is a positive number A such that $\prod_{i=1}^{n}(a_i^s + b_i^s) > A$ for all n. Hence $\{\liminf_{n\to\infty} \prod_{i=k+1}^{n}(a_i^s + b_i^s)\}_{k=0}^{\infty}$ has an upper bound $\dfrac{h^s(F)}{A}$. Using the same argument in the proof of Theorem 1, we have

$$\frac{\mu(B_{\alpha r}(x))}{(\alpha r)^s} \le \frac{\left(\frac{h^s(F)}{A}\right)}{\alpha^{2s}} < \infty.$$

Hence $H^s(F) > \left(\dfrac{A\alpha^{2s}}{h^s(F)}\right) A > 0$. □

Corollary 4 *h^s and H^s are equivalent.*

PROOF. This follows from $H^s \leq h^s$ and Corollary 2 and Theorem 3. □

Example 1 *Let δ be a sufficiently small positive number . We can choose $\epsilon_{k,1}$ and $\epsilon_{k,2}$ for each $k = 1, 2, \cdots$ such that $\frac{-1}{4} + \delta < \epsilon_{k,1}, \epsilon_{k,2} < \frac{1}{4} - \delta$. with the following three cases:*

[Case 1; $(\frac{1}{4} + \epsilon_{k,1})^{\frac{1}{2}} + (\frac{1}{4} + \epsilon_{k,2})^{\frac{1}{2}} = 1$]

Let $a_k = \frac{1}{4} + \epsilon_{k,1}$ and $b_k = \frac{1}{4} + \epsilon_{k,2}$. Then $h^{\frac{1}{2}}(F) = 1$. Using Corollary 4, we see that $0 < H^{\frac{1}{2}}(F) < \infty$.

[Case 2; $(\frac{1}{4} + \epsilon_{k,1})^{\frac{1}{2}} + (\frac{1}{4} + \epsilon_{k,2})^{\frac{1}{2}} = e^{\frac{-1}{k}}$]

Let $a_k = \frac{1}{4} + \epsilon_{k,1}$ and $b_k = \frac{1}{4} + \epsilon_{k,2}$. Then $h^{\frac{1}{2}}(F) = 0$. Using Corollary 4, we see that $H^{\frac{1}{2}}(F) = 0$.

[Case 3; $(\frac{1}{4} + \epsilon_{k,1})^{\frac{1}{2}} + (\frac{1}{4} + \epsilon_{k,2})^{\frac{1}{2}} = e^{\frac{1}{k}}$]

Let $a_k = \frac{1}{4} + \epsilon_{k,1}$ and $b_k = \frac{1}{4} + \epsilon_{k,2}$. Then $h^{\frac{1}{2}}(F) = \infty$. Using Corollary 4, we see that $H^{\frac{1}{2}}(F) = \infty$.

References

[1] I. S. Baek, *Dimensions of the perturbed Cantor set*, Real Analysis Exchange, **19, No.1** (1994), 269–273.

[2] K. J. Falconer, *Fractal Geometry*, John Wiley & Sons. (1990).