Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 603–616

Solomon Leader, Mathematics Department, Hill Center, Busch Campus, Rutgers University, New Brunswick, NJ 08903

TRANSFORMING LEBESGUE-STIELTJES INTEGRALS INTO LEBESGUE INTEGRALS

Abstract

A recent theorem of the author for continuous functions of bounded variation is extended to include the discontinuous case. Given h of bounded variation on a closed interval K let $s(t,y)$ be the total number $0,1,2$ of the following conditions which hold at $(t,y) \in K \times \mathbb{R}$:

- $y = h(t)$,
- y lies strictly between $h(t-)$ and $h(t)$,
- y lies strictly between $h(t)$ and $h(t+)$.

Given f Lebesgue-Stieltjes integrable against $d\,h$ we can define $\hat{f}$ almost everywhere on $\mathbb{R}$ by $\hat{f}(y) = \sum_{t \in K} f(t)\, s(t,y)$ where the nonzero terms form a finite sum. The function $\hat{f}$ is Lebesgue integrable and its integral $\int_{-\infty}^{\infty} \hat{f}(y)\, d\,y = \int_K f\, |d\,h|$. Among the special cases is a generalization of Banach's indicatrix theorem.

1 Introduction

Given a continuous function h on $K = [a,b]$, S. Banach [1] defined its indicatrix $N(y)$ to be the number of points t in K such that $h(t) = y$. He proved that the integral of N exists and equals the total variation of h,

$$(1) \qquad \int_{-\infty}^{\infty} N(y)dy = \int_a^b |dh(t)| \leq \infty.$$

If the continuous function h is of bounded variation then the integrals in (1) are finite so $N < \infty$ almost everywhere. That is, the set $h^{-1}y$ is finite for

Key Words: Lebesgue-Stieltjes integral, Lebesgue integral, bounded variation, regulated function, Banach indicatrix
Mathematical Reviews subject classification: Primary: 26A42 Secondary: 26A45
Received by the editors April 4, 1994

almost all y. In [7] this was used to get for each function f on K a transform φ induced by h under the definition.

$$\varphi(y) = \sum_{t \in h^{-1}y} f(t) \text{ for almost all } y \in \mathbb{R}. \tag{2}$$

If the Lebesgue-Stieltjes integral $\int_K f|dh|$ exists and is finite then φ is Lebesgue - integrable and

$$\int_{-\infty}^{\infty} \varphi(y)dy = \int_a^b f(t)|dh(t)|. \tag{3}$$

This conversion formula is the key to a simple proof of Green's Theorem using a Fubini theorem for the generalized Riemann integral (Theorem 3 in [7].) (3) reduces to Banach's formula (1) if we set $f = 1$ which gives $\varphi = N$ in (2).

Our ultimate objective here is to generalize the transform (2) of f so that the conversion formula(3) applies to all functions h of bounded variation including those with discontinuities. The crucial step is Theorem 1 which shows that for any function h of bounded variation on K Banach's formula (1) generalizes to

$$\int_{-\infty}^{\infty} N(y)dy = \int_a^b 1_C(t)|dh(t)|$$

where 1_C is the indicator of the set C of points of continuity of h.

Geometrically $N(y)$ is the number of intersections of the horizontal line $Y = y$ with the graph of h. If h is discontinuous at t, say $h(t-) \neq h(t)$, then the line $Y = y$ may fail to intersect the graph for y between $h(t-)$ and $h(t)$. In terms of the indicator $q(t, y)$ of the graph $h(t) = y$ the transform (2) of f is

$$\varphi(y) = \sum_{t \in K} f(t)q(t, y) \text{ for almost all } y \tag{4}$$

with summation over the finite set of nonzero terms. If h is discontinuous at t we must add to q the sum $r_- + r_+ = r$ where $r_-(t, y)$ indicates the interior of the line segment joining $(t, h(t-))$ to $(t, h(t))$, and $r_+(t, y)$ indicates the interior of the segment joining $(t, h(t))$ to $(t, h(t+))$. As our final result we shall show (Theorem 4) that (3) holds if q is replaced by $q + r$ in (4).

2 Preliminaries

All integrals here are defined by the Kurzweil-Henstock integration process which we review briefly. For details see [3,4,5].

A <u>cell</u> is a closed, bounded, nondegenerate interval I in $\mathbb{R} = (-\infty, \infty)$. A <u>tagged cell</u> (I, t) is a cell I with one of its endpoints designated as the <u>tag</u> t. A summant S on a cell K is a function $S(I,t)$ on the set of all tagged cells (I,t) in K. Each such S has a lower and upper integral with values in $[-\infty,\infty]$ based on the following definitions. A gauge δ is a function on K with $\delta(t)>0$ for all t. (I,t) is δ-fine if the length of I is less than $\delta(t)$. A division $\mathcal{K}$ of K is a finite set of nonoverlapping tagged cells whose union is K. Such a $\mathcal{K}$ is a δ-division if all its members are δ-fine. Every finite set of nonoverlapping δ-fine tagged cells in K can be extended to a δ-division of K. For S a summant on K each division $\mathcal{K}$ of K yields $\Sigma_{\mathcal{K}}S$, the sum of $S(I,t)$ over the members (I,t) of $\mathcal{K}$. For each gauge δ on K define $\Sigma_\delta S$ to be the infimum, $\Sigma^\delta S$ the supremum, of $\Sigma_{\mathcal{K}}S$ for all δ-divisions $\mathcal{K}$ of K. The lower integral of S is $\underline{\int}_K S = \sup_\delta \Sigma_\delta S$ and the upper integral is $\overline{\int}_K S = \inf_\delta \Sigma^\delta S$ taken over all gauges δ on K. If these two integrals are equal then their common value in $[-\infty,\infty]$ defines the integral $\int_K S$. S is integrable if its integral exists and is finite. Integrability on K implies integrability on every cell contained in K. If S is integrable then either $|S|$ is integrable or $\int_K |S| = \infty$. We call S absolutely integrable if both S and $|S|$ are integrable. A cell summant is a summant whose values do not depend on the tag. An additive summant is a cell summant that is additive on abutting cells. Each function F on K defines an additive summant ΔF given by $\Delta F(I) = F(s) - F(r)$ for each cell $I=[r,s]$ in K. Every additive summant has such a representation. Since $\Sigma_{\mathcal{K}}\Delta F = \Delta F(K)$ for all divisions $\mathcal{K}$ of K, $\int_K \Delta F = \Delta F(K)$. For S a summant and f a function on K the product fS is the summant with value $f(t)S(I,t)$ at (I,t).

From this integration process a sound concept of differential emerges. The equivalence $S \sim S'$ between summants S and S' on K is defined to be $\int_K |S - S'| = 0$. A differential σ on K is the equivalence class $[S]$ of some summant S on K. As a function space the set of all summants on K forms a Riesz space $\mathcal{S}$. The differentials on K form a Riesz space which is the homomorph of $\mathcal{S}$ modulo the Riesz ideal of all summants equivalent to 0. If $\sigma = [S]$ and $\rho = [R]$ then $c\sigma = [cS]$ for any constant c, $\sigma + \rho = [S+R]$, $|\sigma| = [|S|]$, $\sigma^+ = [S^+]$ and $\sigma^- = [S^-]$. The definitions $\underline{\int}_K \sigma = \underline{\int}_K S$ and $\overline{\int}_K \sigma = \overline{\int}_K S$ are effective for any representative S of σ. So we can transfer the definitions of integral, integrability, absolute integrability, etc., to σ from S.

Each function F on K yields an integrable differential $dF = [\Delta F]$ for which $\int_K dF = \Delta F(K)$. A differential σ on K is integrable if and only if $\sigma = dF$ for some function F on K. The total variation of any function F on K is $\int_K |dF| \le \infty$. σ is absolutely integrable if and only if $\sigma = dF$ for some function F of bounded variation on K. A differential σ is summable if $\overline{\int}_K |\sigma| < \infty$. With this upper integral as norm the summable differentials on K form a Banach lattice [5]. For E a subset of K let 1_E be the indicator of E. That is, $1_E(t) = 1$ for t in E, 0 for t in $K-E$. E is σ-null if $1_E S \sim 0$ for $[S] = \sigma$. σ-everywhere means everywhere on $K-E$ for some σ-null E. If

$g(t)$ is defined and finite σ-everywhere on K then the definition $g\sigma = [fS]$ is effective where $\sigma = [S]$ and f is any function on K that equals g σ-everywhere. Thus E is σ-null if and only if $1_E\sigma = 0$. In general, $g = 0$ σ-everywhere if and only if $g\sigma = 0$.

Let x be the identity function $x(t) = t$ on K. Lebesgue-integrability of f is just absolute integrability of $f\ dx$. A subset E of K is Lebesgue-measurable if and only if $1_E\ dx$ is integrable. For such E the Lebesgue measure $M(E)$ is $\int_K 1_E\ dx$.

We shall be concerned here with differentials on K of the form $f\ dh$ and $f|dh|$. $f\ dh$ can be integrable without being absolutely integrable. Moreover $f\ dh$ can be absolutely integrable even though dh is not. This attests to the superiority of Kurzweil-Henstock integration on $\mathbb{R}$ over Lebesgue integration since Lebesgue-Stieltjes integrability of $f\ dh$ demands absolute integrability of dh, $f\ dh$, and $f|dh|$.

3 Regulated functions

Since every function of bounded variation is regulated we shall prove some relevant results on regulated functions. Hereafter h will be a function on $K = [a, b]$, C the set of points at which h is continuous, D the set at which h is discontinuous.

h is regulated if $1_p\ dh$ is integrable for every point p in K. This is equivalent to the usual definition that h has finite unilateral limits at every point p in K, $h(p+)$ for $a \le p < b$ and $h(p-)$ for $a < p \le b$. For convenience we invoke the notational convention that $h(a-) = h(a)$ and $h(b+) = h(b)$ at the endpoints a, b of K. A regulated function h is bounded and has only countably many discontinuities, [2]. For all p in K we have absolute integrability of $1_p\ f\ dh$ for every function f on K. Given p there exists a gauge δ on K such that $\delta(t) < |t - p|$ for $t \ne p$. For such a gauge (I, t) δ-fine with I containing p implies $t = p$. So

$$(5)\qquad \begin{cases} \int_K 1_p\ f(dh)^+ = f(p)([h(p) - h(p-)]^+ + [h(p+) - h(p)]^+) \\ \int_K 1_p\ f(dh)^- = f(p)([h(p) - h(p-)]^- + [h(p+) - h(p)]^-) \\ \int_K 1_p\ f\ dh = f(p)(h(p+) - h(p-)) \\ \int_K 1_p\ f|dh| = f(p)(|h(p) - h(p-)| + |h(p+) - h(p)|). \end{cases}$$

The Monotone Convergence Theorem [3,4] and countability of D give

$$(6)\qquad \begin{cases} \int_K 1_D\ |f|(dh)^+ = \sum_{p\in D} \int_K 1_p\ |f|(dh)^+ \\ \int_K 1_D\ |f|(dh)^- = \sum_{p\in D} \int_K 1_p\ |f|(dh)^- \\ \int_K 1_D\ |f\ dh| = \sum_{p\in D} \int_K 1_p\ |f\ dh|. \end{cases}$$

So for any f on K summability of $1_D\ f\ dh$ is equivalent to absolute integrability.

To get the closure $\overline{hK}$ of hK it suffices to adjoin to hK the unilateral limits $h(t-)$ and $h(t+)$ for t in D. So $\overline{hK}$ is the union of hK with a countable set. Thus hK and its closure have the same Lebesgue measure. Since this holds for every cell I in K we can define for h regulated the cell summant T and its induced differential in terms of Lebesgue Measure M

$$(7)\qquad T(I) = M(hI) \quad \text{and} \quad \tau = [T].$$

We also define the cell summant S and its induced differential,

$$(8)\qquad S(I) = M(\ulcorner hI \urcorner) \quad \text{and} \quad \sigma = [S]$$

where the overbracket denotes convex closure $\ulcorner E \urcorner = [\inf E, \sup E]$. $S(I)$ is just the diameter of hI. If h is continuous then $hI = \ulcorner hI \urcorner$ so $T = S$. But we must investigate T and S in the general case of regulated h.

Hereafter $I \to p$ in K means $M(I) \to 0$ with the cells I in K having p as an endpoint. $I \to p-$ adds the restriction that p be the right endpoint of I, $I \to p+$ that p be the left endpoint.

Lemma 1 *Let h be regulated and σ be defined by (8). Then $1_p\ \sigma = 1_p\ |dh|$ for every point p in K.*

PROOF. Both $S(I)$ and $|\Delta h(I)|$ converge to $|h(p) - h(p-)|$ as $I \to p-$ and to $|h(p+) - h(p)|$ as $I \to p+$. So $\int_K 1_p\ [S - |\Delta h|] = 0$ which proves the lemma since $|\Delta h| \leq S$. □

Lemma 2 *Let h be regulated and $I = [c, d]$ be a cell in K. For each p in K define the closed intervals*

$$(9)\qquad \begin{cases} L_p = \ulcorner h(p), h(p-) \urcorner & \text{if } c < p \leq d, \ \emptyset \text{ otherwise,} \\ R_p = \ulcorner h(p), h(p+) \urcorner & \text{if } c \leq p < d, \ \emptyset \text{ otherwise,} \\ J_p = L_p \cup R_p. \end{cases}$$

Then

$$(10)\qquad \ulcorner hI \urcorner = \bigcup_{p\in I} J_p.$$

PROOF. Let Q be the right side of (10). Clearly $\overline{hI} \subseteq Q \subseteq \ulcorner hI \urcorner$. Suppose (10) false. Then there is some y belonging to $\ulcorner hI \urcorner$ but not to Q. Let A be the set of all p in I such that $h(p) < y$, B the set such that $h(p) > y$. Clearly A and B are disjoint. For all p in I $h(p) \neq y$ since $h(p)$ belongs to J_p but y does not. So $A \cup B = I$. If $t \to p-$ with t in A then p lies in I, $h(t) < y$, and $h(t) \to h(p-)$, so $h(p-) \leq y$. This implies that $h(p-) < y$ since $h(p-)$ belongs to J_p but y does not. So the interval J_p lies in the half-line $(-\infty, y)$. Hence $h(p) < y$ since $h(p)$ belongs to J_p. That is, p belongs to A. Similarly if t belongs to A and $t \to p+$ then p belongs to A. So A is closed. A similar proof affirms that B is closed. So A, B give a topological separation of the connected set I, a contradiction. Thus $Q = \ulcorner hI \urcorner$ giving (10). □

Lemma 3 *If h is regulated and τ is defined by (7) then*

$$1_p \tau = 0. \tag{11}$$

PROOF. Existence of the finite limits $h(p-)$ and $h(p+)$ implies

$$\operatorname{Diam} h(I - p) \to 0 \text{ as } I \to p \in K. \tag{12}$$

Since hI is just $h(I-p)$ united with the single point $h(p)$, $T(I) = M(h(I-p))$ by (7). Thus, since the measure of a set is at most its diameter in $\mathbb{R}$, $T(I) \to 0$ by (12) as $I \to p$. That is, (11) holds. □

Lemma 4 *Let h be regulated and $1_D dh$ be summable. Then $1_D|dh| = dw$ for some function w on K and under (7) and (8)*

$$0 \leq S - T \leq \Delta w, \tag{13}$$

$$1_C \sigma = \tau, \tag{14}$$

$$1_D \sigma = dw, \tag{15}$$

$$\sigma = \tau + dw. \tag{16}$$

PROOF. Summability of $1_D dh$ is just finiteness of (6) for $f = 1$. So $1_D dh$ is absolutely integrable which yields w. Given a cell I in K let $U = \ulcorner hI \urcorner - \overline{hI}$. The interval $\ulcorner hI \urcorner$ has its endpoints in $\overline{hI}$. So U is open in $\mathbb{R}$. From the definition (9) of J_p in Lemma 2 and from (5) for $f = 1$ we have the inequality $M(J_p) \leq \int_I 1_p|dh|$. By Lemma 2 U is covered by the J_p's with p in I. For p in $C \cap I$ J_p consists of the single point $h(p)$ which does not lie in U since U is disjoint from hI. So U is covered by those J_p's with p in $D \cap I$. Therefore $M(U) \leq \sum_{p \in D \cap I} M(J_p) \leq \sum_{p \in D} \int_I 1_p|dh| = \int_I 1_D|dh| = \Delta w(I)$. This together with $M(U) = M(\ulcorner hI \urcorner) - M(hI) = S(I) - T(I)$ gives (13). By

(13) $0 \le \sigma - \tau \le dw$. So $0 \le 1_C(\sigma - \tau) \le 1_C dw = 0$ since C and D are disjoint and $1_D\, dw = dw$. Thus

$$1_C\sigma = 1_C\tau. \tag{17}$$

Since D is countable (11) in Lemma 3 gives $1_D\tau = 0$. So $1_C\tau = \tau$ which with (17) gives (14). By Lemma 1 and the countability of D we get $1_D\sigma = 1_D|dh| = dw$ which gives (15). The sum of (14) and (15) gives (16). □

Lemma 5 *Let h be of bounded variation and $dv = |dh|$ on K. Then under definitions (7) and (8)*

$$\sigma = dv \tag{18}$$

and

$$\tau = 1_C dv. \tag{19}$$

PROOF. (18) follows from $|\Delta h| \le S \le \Delta v$ which implies $dv = |dh| \le \sigma \le dv$. (14) in Lemma 4 and (18) then give (19). □

Lemma 6 *Let h be regulated with $1_D dh$ summable. Then under definition (7)*

$$0 \le \int_K \tau = \int_K 1_C|dh| \le \infty. \tag{20}$$

PROOF. Since $|\Delta h| \le S$ by (8), $|dh| \le \sigma$. Hence $1_C|dh| \le 1_C\sigma$. By (14) in Lemma 4 this is just

$$1_C|dh| \le \tau. \tag{21}$$

Now $1_C|dh| = |dh| - 1_D|dh|$ with $1_D|dh|$ integrable. Thus since the integral of $|dh|$ exists so does

$$\int_K 1_C|dh| = \int_K |dh| - \int_K 1_D|dh| \le \infty. \tag{22}$$

If h is of bounded variation then (20) follows from (19) in Lemma 5. If h is of unbounded variation then (22) is infinite which by (21) gives (20) with both integrals infinite. □

We can now extend the role of Banach's indicatrix N. Note that the hypothesis of Theorem 1 is satisfied if h is continuous.

Theorem 1 *Let h be regulated on K with $1_D dh$ summable. Then for $N(y)$ the number of points t such that $h(t) = y$*

$$\int_{-\infty}^{\infty} N(y)dy = \int_K 1_C|dh| \le \infty. \tag{23}$$

PROOF. Using (20) in Lemma 6 we can get a sequence of gauges δ_n on K such that

$$\delta_n < 1/n \tag{24}$$

and for every δ_n-division $\mathcal{K}$ of K

$$\begin{cases} |\sum_{\mathcal{K}} T - \int_K 1_C|dh|| < 1/n \ \text{ if } \ \int_K 1_C|dh| < \infty \\ \sum_{\mathcal{K}} T > n \ \text{ if } \ \int_K 1_C|dh| = \infty \end{cases} \tag{25}$$

where T is defined by (7). Choose a sequence of divisions $\mathcal{K}_n$ such that

$$\mathcal{K}_n \ \text{ is a } \delta_n\text{-division of } \ K \tag{26}$$

and

$$\boldsymbol{K}_{n+1} \ \text{ refines } \ \boldsymbol{K}_n \tag{27}$$

where $\boldsymbol{K}_n$ is the partition induced by $\mathcal{K}_n$. (That is, I belongs to $\boldsymbol{K}_n$ if and only if (I, t) belongs to $\mathcal{K}_n$ for some t.) For each y in $\mathbb{R}$ let $N_n(y)$ be the number of members I of $\boldsymbol{K}_n$ such that y belongs to hI. That is,

$$N_n(y) = \sum_{I \in \boldsymbol{K}_n} 1_{hI}(y). \tag{28}$$

By (27) and (28)

$$0 \leq N_n(y) \leq N_{n+1}(y) < \infty \text{ for all } y \text{ in } \mathbb{R} \text{ and all } n. \tag{29}$$

Let E be the set of endpoints of the cells belonging to $\boldsymbol{K}_1 \cup \boldsymbol{K}_2 \cup \cdots$. Since E is countable so is hE. Consider any y that does not belong to hE. Given t in $h^{-1}y$ each $\boldsymbol{K}_n$ has just one member I that contains t since t must be interior to I and the members of a partition do not overlap. Thus by (28)

$$N_n(y) \leq N(y) \leq \infty \text{ for all } n, \text{ and all } y \text{ in the complement of the countable set } hE. \tag{30}$$

Consider any finite subset A of $h^{-1}y$. Take n large enough to make $1/n$ less than the distance between any pair of distinct points in A. For such n (24) and (26) imply that each member I of $\boldsymbol{K}_n$ contains at most one point of A. This together with (29) and (30) gives

$$N_n(y) \nearrow N(y) \text{ for almost all } y \text{ as } n \nearrow \infty \tag{31}$$

since the countable set hE is of measure zero. Now $\int_{-\infty}^{\infty} 1_{hI}(y)dy = M(hI) = T(I)$. So integration of (28) gives

$$\int_{-\infty}^{\infty} N_n(y)dy = \sum_{\mathcal{K}_n} T. \tag{32}$$

By the Monotone Convergence Theorem applied to (31) the integral in (32) converges to $\int_{-\infty}^{\infty} N(y)dy$. On the other hand the right side of (32) converges to $\int_K 1_C|dh|$ by (25) and (26). This gives (23). □

4 The conversion formulas for integrators h of bounded variation

Theorem 2 *Let h be of bounded variation on K with $dv = |dh|$. Given a function f on K define the function ϕ almost everywhere or $\mathbb{R}$ by*

$$\phi(y) = \sum_{t \in h^{-1}y} f(t). \tag{33}$$

If $f\ dv$ is absolutely integrable then ϕ is Lebesgue-integrable and

$$\int_{-\infty}^{\infty} \phi(y)dy = \int_K 1_C f\ dv. \tag{34}$$

PROOF. *Case 1.* $0 \le f \le k$ for some integer k.

Since C is a Borel set $1_C dv$ is integrable. So there is a function w on K such that

$$dw = 1_C dv = 1_C|dh|. \tag{35}$$

w is monotone, $dw \ge 0$. w is continuous since $1_p dw = 0$ for every point p in K. Similarly since $1_C f\ dv$ is integrable there is a function F on K such that

$$dF = 1_C f\ dv = f\ dw. \tag{36}$$

ϕ is defined almost everywhere by (33) since $N < \infty$ almost everywhere by Theorem 1. To prove (34) we must show in terms of (36) that

$$\int_{-\infty}^{\infty} \phi(y)dy = \Delta F(K). \tag{37}$$

Under Case 1 (36) gives $0 \le dF \le k\ dw$. Integration over any cell in K gives the corresponding summant inequality

$$0 \le \Delta F \le k\ \Delta w. \tag{38}$$

Define the cell summant G by

$$G(I) = \frac{\Delta F}{\Delta w}(I) \text{ if } \Delta w(I) > 0,\ 0 \text{ if } \Delta w(I) = 0 \tag{39}$$

for every cell I in K. From (39) and (38) we get

$$G\,\Delta w = \Delta F \tag{40}$$

and

$$0 \le G \le k. \tag{41}$$

By (19) in Lemma 5 (35) is just $T \sim \Delta w$ under definition (7). So $GT \sim G\,\Delta w$ by the boundedness (41) of G. By (40) this is just $GT \sim \Delta F$. So we can choose for each positive integer n a gauge δ_n on K such that $\delta_n < 1/n$ and

$$\sum^{\delta_n} |\Delta F - GT| < 1/n. \tag{42}$$

Choose a sequence of δ_n-divisions $\mathcal{K}_n$ of K such that $\boldsymbol{K}_{n+1}$ refines $\boldsymbol{K}_n$ where $\boldsymbol{K}_n$ is the partition induced by $\mathcal{K}_n$. For each n define the function ϕ_n on $\mathbb{R}$ by

$$\phi_n(y) = \sum_{I \in \boldsymbol{K}_n} G(I) 1_{hI}(y). \tag{43}$$

Each ϕ_n is a linear combination of indicators of bounded measurable sets hI. So ϕ_n is Lebesgue-integrable and integration of (43) gives

$$\int_{-\infty}^{\infty} \phi_n(y)dy = \sum_{\mathcal{K}_n} GT. \tag{44}$$

Since $\mathcal{K}_n$ is a δ_n-division (42) and (44) give the convergence

$$\int_{-\infty}^{\infty} \phi_n(y)dy \to \Delta F(K) \text{ as } n \to \infty. \tag{45}$$

Let B be the set of all t in C where the condition

$$\frac{dF}{dw}(t) = f(t) \tag{46}$$

fails to hold. B is dw-null by Theorem 17 in [4]. So $0 = 1_B dw = 1_B|dh|$ by (35) since B is a subset of C. That is, B is dh-null. So hB is Lebesgue-null by Theorem 2 in [6]. So is $N^{-1}\infty$ by (23) in Theorem 1. D is countable as is the set E of endpoints of members of $\boldsymbol{K}_1 \cup \boldsymbol{K}_2 \cup \cdots$. Dismissing the Lebesgue-null sets hB, $N^{-1}\infty$, hD, and hE we conclude that for almost all y

$$\text{(47)} \qquad \left\{ \begin{array}{l} h^{-1}y \text{ is a finite subset of } C - B \text{ covered for each} \\ n \text{ by the interiors of the cells belonging to } \boldsymbol{K}_n. \end{array} \right.$$

Consider any y in hK satisfying (47). $h^{-1}y$ is then a nonvoid finite set $\{t_1, \ldots, t_m\}$. Since $\mathcal{K}_n$ is a $1/n$-division it follows that for n sufficiently large each member I of $\boldsymbol{K}_n$ contains at most one point of $h^{-1}y$. For such n (43) takes the form

$$\text{(48)} \qquad \phi_n(y) = \sum_{i=1}^{m} G(I_{i,n})$$

where $I_{i,n}$ is the unique member of $\boldsymbol{K}_n$ whose interior contains t_i. Since (46) holds on $h^{-1}y$, a subset of $C - B$ by (47), the definition (39) of G together with the condition that $M(I_{i,n}) < 1/n$ gives the convergence

$$\text{(49)} \qquad G(I_{i,n}) \to f(t_i) \text{ as } n \to \infty$$

for $i = 1, \ldots, m$. By (33), (48) and (49)

$$\text{(50)} \qquad \phi_n(y) \to \phi(y) \text{ as } n \to \infty$$

for all y satisfying (47), hence almost everywhere. For y subject to the coverage condition in (47) the definition (43) of ϕ_n and the boundedness (41) of G give

$$\text{(51)} \qquad 0 \le \phi_n(y) \le k\, N(y) \text{ for all } n.$$

So (51) holds for almost all y. Since h is of bounded variation, N is Lebesgue-integrable by Theorem 1. So (50), (51) and (45) give (37) by the Dominated Convergence Theorem. So Theorem 2 holds for Case 1.

Case 2. $0 \le f(t) < \infty$ for all t in K. Apply Case 1 to $f_k = k \wedge f$ for $k = 1, 2, \ldots$. With w satisfying (35) the Monotone Convergence Theorem gives for $f_k \nearrow f$

$$\text{(52)} \qquad \int_K f_k dw \nearrow \int_K f\, dw \text{ as } k \nearrow \infty.$$

Define ϕ_k almost everywhere by

$$\text{(53)} \qquad \phi_k(y) = \sum_{t \in h^{-1}y} f_k(t)$$

which is just (33) with f replaced by f_k. Applying Case 1 to f_k we get (34) for f_k, namely

$$\int_{-\infty}^{\infty} \phi_k(y)dy = \int_K f_k dw. \tag{54}$$

Since $f_k \nearrow f$, $\phi_k \nearrow \phi$ a.e. by (53) and (33). So the Monotone Convergence Theorem gives

$$\int_{-\infty}^{\infty} \phi_k(y)dy \nearrow \int_{\infty}^{\infty} \phi(y)dy \text{ as } k \nearrow \infty. \tag{55}$$

From (52), (54), (55) we conclude that $\int_{-\infty}^{\infty} \phi(y)dy = \int_K f\, dw$ which is just (34) for Case 2. Note that our proof also shows that for any Borel function $f \geq 0$ with $\int_K f\, dw = \infty$ (34) holds with both integrals infinite.

Case 3. $-\infty < f(t) < \infty$ for all t in K. Apply Case 2 to both f^+ and f^- and subtract the results to get the theorem for $f = f^+ - f^-$. □

Note that in Theorem 2 we can replace f by $1_C f$. So the hypothesis on f is that $1_C f|dh|$ is absolutely integrable.

We turn now to the easily proved counterpart of Theorem 2 where C is replaced by its complement D.

Theorem 3 *Given h regulated on K let $r = r_- + r_+$ on $K \times \mathbb{R}$ where $r_-(t,y)$ indicates that y lies strictly between $h(t-)$ and $h(t)$, and $r_+(t,y)$ indicates that y lies strictly between $h(t)$ and $h(t+)$. Given f on K such that $1_D f\, dh$ is summable the function θ is defined almost everywhere on $\mathbb{R}$ by*

$$\theta(y) = \sum_{t \in D} f(t) r(t,y) \tag{56}$$

since the series is absolutely convergent for almost every y. Moreover, θ is Lebesgue integrable and

$$\int_{-\infty}^{\infty} \theta(y)dy = \int_K 1_D f|dh|. \tag{57}$$

PROOF. For each point p in D we have

$$\int_{-\infty}^{\infty} f(p)r(p,y)dy = f(p)[\int_{-\infty}^{\infty} r_-(p,y)dy + \int_{-\infty}^{\infty} r_+(p,y)dy]$$
$$= f(p)[|h(p) - h(p-)| + |h(p+) - h(p)|] = \int_K 1_p f|dh|$$

by the last equation in (5). Apply (5) and (6) to both f^+ and f^- to get (57) and absolute convergence almost everywhere of the series in (56) from the Monotone Convergence Theorem. This also shows that (57) holds under (56) for all $f \geq 0$ on K, allowing infinite values in (56) and (57). □

We can now combine the results of Theorems 2 and 3.

Theorem 4 *Let h be of bounded variation on K with $dv = |dh|$. Let $q(t,y)$ indicate in $K \times \mathbb{R}$ that $h(t) = y$. Let $s = q + r$ with r as defined in Theorem 3. Let f be a function on K with $f\,dv$ absolutely integrable. Then the transform $\hat{f}$ of f is defined almost everywhere on $\mathbb{R}$ by*

$$\hat{f}(y) = \sum_{t \in K} f(t)s(t,y) \tag{58}$$

with the nonzero terms forming a finite sum for almost all y. Moreover $\hat{f}$ is Lebesgue-integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} \hat{f}(y)dy = \int_K f\,dv. \tag{59}$$

PROOF. Apply Theorems 2 and 3. Note that the definition (33) of ϕ is just (4). So (58) is the sum of (33) and (56) with $\hat{f} = \phi + \theta$. (59) is the sum of (34) and (57) since $1_C + 1_D = 1$. □

Note that $s(t,y) = 0$ for y in the complement of $\ulcorner \overline{hK} \urcorner$. So $\hat{f}(y) = 0$ for such y by (58). So the integral of $\hat{f}$ in (59) may be taken over the bounded interval $\ulcorner \overline{hK} \urcorner$ instead of $\mathbb{R}$.

Except for y in hD we have $h^{-1}y$ contained in C which implies $q(t,y) = 1_{h^{-1}y}(t) \leq 1_C(t)$ so $1_C(t)q(t,y) = q(t,y)$. We also have $1_C(t)r(t,y) = 0$ by the definition of r in Theorem 3. So $1_C(t)s(t,y) = 1_C(t)q(t,y) + 1_C(t)r(t,y) = q(t,y)$ except for y in the countable set hD. Thus (58) and (33) give $\widehat{1_C f}(y) = \sum_{t \in K} f(t)q(t,y) = \phi(y)$ for all but countably many y. So $\widehat{1_C f} = \emptyset$ almost everywhere which reduces (59) to (34). So Theorem 2 is the special case of Theorem 4 for $f = 1_C f$.

Except for y in hD we have $1_D(t)q(t,y) = 0$. Also $1_D(t)r(t,y) = r(t,y)$. So $1_D(t)s(t,y) = r(t,y)$ except for y in hD. Thus (58) and (56) give $\widehat{1_D f}(y) = \sum_{t \in K} f(t)1_D(t)s(t,y) = \sum_{t \in D} f(t)r(t,y) = \theta(y)$ for all but countably many y. So $\widehat{1_D f} = \theta$ almost everywhere which reduces (59) to (57). So for $f = 1_D f$ Theorem 4 reduces to Theorem 3 for h of bounded variation.

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