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THE RIGHT ABSORPTION PROPERTY FOR DARBOUX FUNCTIONS

Abstract

In this paper we investigate the Darboux right absorption property of a family of function. In particular, the considerations concentrate around the maximal class of functions having the Darboux right absorption property.

In paper [CN] the authors proved (Theorem 1.1): Let f be a continuous function from the real line $\mathbb{R}$ onto $\mathbb{R}$. If g is a function from $\mathbb{R}$ into $\mathbb{R}$ such that $g \circ f$ is continuous, then g is continuous. This theorem leads directly to the definition ([CN]): Let $\mathcal{F}$ be a class of functions from a space X into itself, such that at least one element from $\mathcal{F}$ is a surjection. We say that $\mathcal{F}$ has the right absorption property (abbreviated RAP) provided that if $g : X \to X$ is such that $g \circ f \in \mathcal{F}$ for some surjection $f \in \mathcal{F}$, then $g \in \mathcal{F}$.

The last example of paper [CN] shows that the family D of all Darboux functions does not possess RAP. In this paper we analyse the problem of the right absorption property for Darboux functions more precisely (we assume that $h : \mathbb{R} \to Y$, where Y is some topological space, is a Darboux function if the image of an arbitrary closed interval is a connected set).

First, we shall modify the definition from paper [CN]: Let $\mathcal{F}$ be some family of Darboux surjections mapping $\mathbb{R}$ onto $\mathbb{R}$. We say that $\mathcal{F}$ has the Darboux right absorption property, relative to a topological space Z (abbreviated $DRAP(Z)$) , provided that if $g : \mathbb{R} \to Z$ is such that $g \circ f$ is a Darboux function for some $f \in \mathcal{F}$, then g is also a Darboux transformation.

In the whole paper we consider only a perfectly normal topological space ([ER]): A topological space X is called a perfectly normal space ($T_6 - space$) if X is a normal space and every closed subset of X is a G_δ set.

Let $f : \mathbb{R} \xrightarrow{\text{onto}} Y$ where Y is a topological space. We say that f is a bilateral uniformly discontinuous function at $\alpha \in Y$ if there exists a neighbourhood V_α

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of α such that $f([x, x+\eta))\setminus V_\alpha \neq \emptyset \neq f((x-\eta, x])\setminus V_\alpha$ for each $x \in f^{-1}(\alpha)$ and each $\eta > 0$.

To simplify the notation, we denote by D_Y^* the family of all Darboux functions which possess no point of bilateral uniform discontinuity .

A net in a topological space X is an arbitrary function from a nonempty directed set to the space X. Nets will be denoted by the symbol $\{x_\sigma\}_{\sigma\in\Sigma}$, where x_σ is the point of X assigned to the element σ of the directed set Σ. We shall apply the notation connected with the nets introduced in [ER]. Moreover, in a topological space X, for a net $\{x_\sigma\}_{\sigma\in\Sigma}$, let $\operatorname{acp}_{\sigma\in\Sigma} x_\sigma$ denote the set of all accumulation points of $\{x_\sigma\}_{\sigma\in\Sigma}$.

By $\mathbb{N}$ we denote the set of all positive integers.

Let us remark that if $\mathcal{F}$ is the family of all linear functions $f : \mathbb{R} \to \mathbb{R}$, then $\mathcal{F}$ has $DRAP(\mathbb{R})$. At the same time, it is easy to construct a nonlinear function g such that $\mathcal{F} \cup \{g\}$ has $DRAP(\mathbb{R})$. On the other hand, the example from paper [CN] shows that the class of all Darboux surjections does not possess $DRAP(\mathbb{R})$. So, the questions connected with the maximal class $\mathcal{F}$ which possesses $DRAP(Z)$ are interesting[1] (in the sequel, this family will always be denoted by $\mathcal{F}_D$).

Of course, one can find some analogies between this subject and the problems of the maximal additive and multiplicative class for Darboux functions. It should be noted here that the investigations connected with the seeking for the maximal additive or multiplicative family for Darboux functions allowed to obtain many interesting and elegant mathematical results (e.g. [RT], [FR], [BA], [BC]).

The first considerations suggested that trying to find the family $\mathcal{F}_D$ necessitates concentrating on the continuity of the transformation. A more thorough analysis of the problem allowed us to distinguish the class $\hat{C}$ of transformations.

Definition 1 *We say that a Darboux surjection* $f : \mathbb{R} \xrightarrow{onto} \mathbb{R}$ *belongs to the family* $\hat{C}$ *if, for an arbitrary real number* α, *there exist real numbers* x_α, y_α *such that* $f|_{[x_\alpha, y_\alpha]}$ *is a continuous function and* $\alpha \in \operatorname{Int} f([x_\alpha, y_\alpha])$.

Then the questions (relative to a fixed topological space Z), announced earlier, will take the form:

1. Does $\hat{C}$ possess $DRAP(Z)$?

[1] We can assume a definition similar to that of the maximal additive and multiplicative class of functions ([BA], [BC]): The family $\mathcal{F}_D$ of Darboux surjections will be called the maximal class possessing $DRAP(Z)$ if $\mathcal{F}_D$ has $DRAP(Z)$, and for each family K, which possesses $DRAP(Z)$, we have $K \subset \mathcal{F}_D$. Of course, $\mathcal{F}_D$ is the class of all Darboux surjections f such that if the superposition $g \circ f$ is a Darboux function, then g is a Darboux function, too.

In the case of the positive answer:

2. Is $\hat{C}$ the maximal class possessing $DRAP(Z)$?

   In the case when the answer to the last question is "no":

3. Characterize the maximal class having $DRAP(Z)$.

   An easier version of this problem is also possible:

3' Show a (pretty small) family of transformations in which a class possessing $DRAP(Z)$ is contained.

Theorem 1 contains the answers to questions 1,2 and 3'. Question 3 is an open problem.

Before we formulate this theorem, we shall prove two lemmas.

Lemma 1 *Let $f \in \hat{C}$ and $\alpha_0 \in \mathbb{R}$. Then there exist points*[2] $x_0, y_0 \in \mathbb{R}$ *and a real number $\delta_0 > 0$, such that $f(x_0) = \alpha_0 = f(y_0)$, the functions $f|_{[x_0-\delta_0, x_0+\delta_0]}$, $f|_{[y_0-\delta_0, y_0+\delta_0]}$ are continuous and, moreover,*

$$f([x_0-\delta_0,\, x_0)) \subset (\alpha_0, +\infty) \qquad \text{or} \qquad f((x_0,\, x_0+\delta_0]) \subset (\alpha_0, +\infty)$$

and

$$f([y_0-\delta_0,\, y_0)) \subset (-\infty, \alpha_0) \qquad \text{or} \qquad f((y_0,\, y_0+\delta_0]) \subset (-\infty, \alpha_0).$$

PROOF. Let $\alpha_0 \in \mathbb{R}$ and let $x_{\alpha_0}, y_{\alpha_0}$ be real numbers such that $g = f|_{[x_{\alpha_0}, y_{\alpha_0}]}$ is a continuous function and $\alpha_0 \in Int(f([x_{\alpha_0}, y_{\alpha_0}]))$. Then $f([x_{\alpha_0}, y_{\alpha_0}]) \cap (\alpha_0, +\infty) \neq \emptyset$. Put $A = g^{-1}((\alpha_0, +\infty))$ and let $z \in A$. Then there exists $\hat{x} \in [x_{\alpha_0}, y_{\alpha_0}]$ such that $f(\hat{x}) < \alpha_0$. Suppose, for instance, that $\hat{x} < z$. Hence $z \neq x_{\alpha_0}$, and so, $\{x \in [x_{\alpha_0}, y_{\alpha_0}] : x < z \wedge (x, z) \subset A\} \neq \emptyset$. Denote $x_0 = \inf\{x \in [x_{\alpha_0}, y_{\alpha_0}] : x < z \wedge (x, z) \subset A\}$ and $\delta_0^{x_0} = \min(|z - x_0|, |x_0 - x_{\alpha_0}|) > 0$. Then $f|_{[x_0-\delta_0^{x_0}, x_0+\delta_0^{x_0}]}$ is a continuous function and $f((x_0,\, x_0+\delta_0^{x_0}]) \subset (\alpha_0, +\infty)$.

We remark that $f(x_0) = \alpha_0$. Indeed, according to the definition of x_0, we deduce that there exists a sequence $\{x_n\} \subset A$ such that $x_n \searrow x_0$. Then $f(x_0) \in [\alpha_0, +\infty)$ and $x_0 \notin A$, which proves that $f(x_0) = \alpha_0$.

In a similar way we define y_0 and $\delta_0^{y_0}$. To finish the proof of this lemma, it suffices to put $\delta_0 = \min(\delta_0^{x_0}, \delta_0^{y_0})$. □

The next lemma is a partial answer to the question connected with the properties of functions belonging to $\mathcal{F}_D$.

[2] Of course, the points x_0, y_0 need not be distinct.

Lemma 2 *Let $f : \mathbb{R} \xrightarrow{onto} \mathbb{R}$ be a function such that, for some $\alpha_0 \in \mathbb{R}$, there exist* [3] *$x_0, y_0 \in \mathbb{R}$ for which $f(x_0) = \alpha_0 = f(y_0)$ and there exist $\delta_0^{x_0}, \delta_0^{y_0} > 0$ such that:*

- *$f|_{[x_0-\delta_0^{x_0}, x_0]}$ is a continuous function and $f([x_0 - \delta_0^{x_0}, x_0)) \subset (\alpha_0, +\infty)$ or*
- *$f|_{[x_0, x_0+\delta_0^{x_0}]}$ is a continuous function and $f((x_0, x_0 + \delta_0^{x_0}]) \subset (\alpha_0, +\infty)$ and, moreover,*
- *$f|_{[y_0-\delta_0^{y_0}, y_0]}$ is a continuous function and $f([y_0 - \delta_0^{y_0}, y_0)) \subset (-\infty, \alpha_0)$ or*
- *$f|_{[y_0, y_0+\delta_0^{y_0}]}$ is a continuous function and $f((y_0, y_0 + \delta_0^{y_0}]) \subset (-\infty, \alpha_0)$.*

Then, for each function $g : \mathbb{R} \to Z$ where Z is some topological space, if x_0, y_0 are Darboux points of the first kind [4] *of $g \circ f$, then α_0 is a Darboux point of the the first kind of g.*

PROOF. According to our assumption that Z is a perfectly normal space, it is sufficient to show that α_0 is a Darboux point of the the second kind of g.

So, let $L = [\alpha_0, \alpha_0']$ be an arbitrary segment with endpoint at α_0 (suppose, for instance, that $\alpha_0' > \alpha_0$).

Assume that

$$\overline{g([\alpha_0, \beta])} = Z \qquad \text{for each} \quad \beta \in L\backslash\{\alpha_0\}. \tag{1}$$

By the assumption of the lemma, there exist a point x_0 and a real number $\delta_0^{x_0} > 0$, such that $f(x_0) = \alpha_0$ and the condition connected with the

[3] Of course, it is possible that x_0, y_0 are equal to each other.

[4] The definitions of Darboux points of the first and second kinds are contained in papers ([RJP], Definition 4.1, p. 42) and [RP], Definition 1.2): We say that a point $x_0 \in X$ is a Darboux point of the first kind (of $f : X \to Y$) if for every arc L $= L(x_0, a)$ (with endpoints at x_0 and a) the following conditions are fulfilled:

- If $\overline{f(L_{\mathrm{L}}(x_0, p))} = Y$ ($L_{\mathrm{L}}(x_0, p)$ denote subarc of L with endpoints at x_0 and p) for every element $p \in \mathrm{L} \setminus \{x_0\}$, then there exists a point $p_0 \in \mathrm{L} \setminus \{x_0\}$ such that $f(L_{\mathrm{L}}(x_0, p_0))$ is a connected set.
- If K is a set such that for some net $\{x_\sigma\}_{\sigma\in\Sigma} \subset \mathrm{L}$ for which $x_0 \in \lim_{\sigma\in\Sigma} x_\sigma$, K quasi-cuts $f(\mathrm{L}) \cup \mathrm{acp}_{\sigma\in\Sigma} f(x_\sigma)$ between the sets $\{f(x_0)\}$ and $\{f(x_\sigma) : \sigma \in \Sigma\} \cup \mathrm{acp}_{\sigma\in\Sigma} f(x_\sigma)$, then $K \cap f(L_{\mathrm{L}}(x_0, x_\sigma)) \neq \emptyset$, for every $\sigma \in \Sigma$.
- If for some net $\{x_\sigma\}_{\sigma\in\Sigma} \subset \mathrm{L}$ for which $x_0 \in \lim_{\sigma\in\Sigma} x_\sigma$, $Y \setminus f(\mathrm{L})$ quasi-cuts $f(\mathrm{L})$ into sets A and B between the sets $\{f(x_0)\}$ and $\{\underline{f(x_\sigma)} : \sigma \in \Sigma\}$ in such a way that $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A} \cap \overline{B}$ is of type G_δ in a subspace $\overline{A \cup B}$ of Y.

A point $x_0 \in X$ is a Darboux point of the second kind (of f) if for every arc L $= L(x_0, a)$ conditions (1) and (2) are fulfilled.

half-line $(\alpha_0, +\infty)$ is fulfilled. Suppose, for instance, that $h = f|_{[x_0-\delta_0^{x_0}, x_0]}$ is a continuous function and $f([x_0 - \delta_0^{x_0}, x_0)) \subset (\alpha_0, +\infty)$. Then there exists a point $x_0' \in [x_0 - \delta_0^{x_0}, x_0)$ such that $h([x_0', x_0]) \subset L$. From the fact that $f([x_0', x_0])$ is a nondegenerate interval included in L with endpoint at α_0 and according to (1) we may infer that $\overline{(g \circ f)([x_0', x_0])} = Z$. Since x_0 is a Darboux point of the the first kind of $g \circ f$, there exists $z_0 \in [x_0', x_0]$ such that $(g \circ f)([z_0, x_0])$ is a connected set. Let $\alpha_0^* = \sup(f([z_0, x_0])) \in L \backslash \{\alpha_0\}$. Then $g([\alpha_0, \alpha_0^*]) = g(f([z_0, x_0]))$ is a connected set, too.

To finish the proof of this lemma, it suffices to show that:

if K is a set such that, for some net $\{\alpha_\sigma\}_{\sigma\in\Sigma} \subset L$ for which $\alpha_0 = \lim_{\sigma\in\Sigma} \alpha_\sigma, K$ quasi-cuts $g(L) \cup \mathrm{acp}_{\sigma\in\Sigma}\, g(\alpha_\sigma)$ between the sets $\{g(\alpha_0)\}$ and $\{g(\alpha_\sigma) : \sigma \in \Sigma\} \cup \mathrm{acp}_{\sigma\in\Sigma}\, g(\alpha_\sigma)$, then $K \cap g([\alpha_0, \alpha_\sigma]) \neq \emptyset$ for any $\sigma \in \Sigma$.

Let x_0, x_0', δ^{x_0} and h be the two points, the positive real number and the function from the first part of the proof. Without loss of generality we may assume that $\{\alpha_\sigma : \sigma \in \Sigma\} \subset (f([x_0', x_0])$. Let $\ll$ be the relation directing Σ.

Let us define

$$\Delta = \left\{\left(\sigma, (x_0 - \frac{1}{n}, x_0]\right) : \sigma \in \Sigma \wedge n \in \mathbb{N} \wedge \alpha_\sigma \in f\left((x_0 - \frac{1}{n}, x_0]\right)\right\}.$$

At present we define the relation $\preceq$ in Δ in the following way:

$$\left(\sigma_1, (x_0 - \frac{1}{n_1}, x_0]\right) \preceq \left(\sigma_2, (x_0 - \frac{1}{n_2}, x_0]\right) \Leftrightarrow \sigma_1 \ll \sigma_2 \wedge n_2 \geq n_1.$$

Observe that $\preceq$ directs Δ. Now, we define a net $\{\beta_\delta\}_{\delta\in\Delta}$ by the sentence:

for each $\delta = \left(\sigma, (x_0 - \frac{1}{n}, x_0]\right) \in \Delta$, let β_δ be an arbitrary element of the set $f^{-1}(\alpha_\sigma) \cap \left(x_0 - \frac{1}{n}, x_0\right]$.

Then $x_0 = \lim_{\delta\in\Delta} \beta_\delta$, and $\{f(\beta_\delta)\}_{\delta\in\Delta}$ is a subnet of $\{\alpha_\sigma\}_{\sigma\in\Sigma}$, and so, $\alpha_0 = \lim_{\delta\in\Delta} f(\beta_\delta)$. Observe that $\{g(f(\beta_\delta))\}_{\delta\in\Delta}$ is a subnet of $\{g(\alpha_\sigma)\}_{\sigma\in\Sigma}$, which means that $\mathrm{acp}_{\delta\in\Delta}\, g(f(\beta_\delta)) \subset \mathrm{acp}_{\sigma\in\Sigma}\, g(\alpha_\sigma)$. Consequently, K quasi-cuts $g(L) \cup ak\, \mathrm{acp}_{\delta\in\Delta}\, g(f(\beta_\delta))$ between the sets $\{g(f(x_0))\}$ and $\{g(f(\beta_\delta)) : \delta \in \Delta\} \cup \mathrm{acp}_{\delta\in\Delta}\, g(f(\beta_\delta))$. Since x_0 is a Darboux point of the the first kind of $g \circ f$, therefore

$$K \cap g(f([\beta_\delta, x_0])) \neq \emptyset \qquad \text{for each} \quad \delta \in \Delta,$$

and so,

$$K \cap g([\alpha_0, \alpha_\sigma]) \neq \emptyset \qquad \text{for each} \quad \sigma \in \Sigma.$$

□

Theorem 1 *Let Z be a nonsingleton connected space such that there exists a continuous surjection $h : \mathbb{R} \to Z$. Then*

$$\hat{C} \subset \mathcal{F}_D \subset D_Z^* \qquad \text{and} \qquad \hat{C} \neq \mathcal{F}_D \neq D_Z^*.$$

PROOF. According to Lemmas 2 and 1, we may infer ([RP]) that the inclusion $\hat{C} \subset \mathcal{F}_D$ takes place[5].

Now, we shall show that $\hat{C} \neq \mathcal{F}_D$.

Let $t_{-1} : (-\infty, 0] \to \mathbb{R}$ be defined by the formula $t_{-1}(x) = x$. A function $t_0' : \left(0, \frac{1}{2}\right] \to \mathbb{R}$ is defined in the following way: $t_0'(x) = \max\left(-x^2 + x, \frac{1}{x}|\sin\frac{1}{x}|\right)$. Now, a function $t_0 : [0,1] \to \mathbb{R}$ may be defined as follows:

$$t_0(x) = \begin{cases} 0 & \text{for } x \in \{0,1\}, \\ t_0'(x) & \text{for } x \in \left(0, \frac{1}{2}\right], \\ t_0'(1-x) & \text{for } x \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Of course, $t_0|_{(0,1)}$ is a continuous function, t_0 is a Darboux function and $t_0([0,1]) = [0, +\infty)$. Suppose that we have defined functions $t_i : [i, i+1] \to \mathbb{R}$ for $i = 0, 1, 2, \ldots, n$. Then let $t_{n+1} : [n+1, n+2] \to \mathbb{R}$ be defined by the formula

$$t_{n+1}(x) = \min\left(\frac{1}{n+1}, t_0(x-n-1)\right).$$

Put $t = \nabla_{n=-1}^{\infty} t_n : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ (see [ER], p. 99).

Now, we shall show that if $g : \mathbb{R} \to Z$ is a function such that $g \circ t$ is a Darboux transformation, then g is a Darboux function, too. By Lemma 2 and according to the construction of the functions t_{-1} and t_0', we may deduce that if $\alpha \in \mathbb{R} \setminus \{0\}$, then α is a Darboux point of the the first kind of g. To finish the proof of the fact that g is a Darboux function, it suffices to show that 0 is a Darboux point of the the first kind (and, according to the perfect normality of Z, that 0 is a Darboux point of the the second kind) of g.

Let γ be an arbitrary real number different from 0. Denote $L = [0, \gamma]$.

At present, we suppose that $\overline{g([0,\beta])} = Z$ for each $\beta \in L \setminus \{0\}$. If $\gamma < 0$, then, similarly as in the proof of Lemma 2, we can show that there exists $\gamma_0 \in L \setminus \{0\}$ such that $g([\gamma_0, 0])$ is a connected set. So, let $\gamma > 0$. Fix $n_\gamma \in \mathbb{N}$ such that $\frac{1}{n_\gamma} < \gamma$. Then $g\left(\left[0, \frac{1}{n_\gamma}\right]\right) = (g \circ t)\left(\left[n_\gamma, n_\gamma + \frac{1}{2}\right]\right)$ is a connected set.

Now, we suppose that K is a set such that, for some net $\{\alpha_\sigma\}_{\sigma \in \Sigma} \subset [0, \gamma]$ such that $0 = \lim_{\sigma \in \Sigma} \alpha_\sigma$, K quasi-cuts $g([0,\gamma]) \cup \operatorname{acp}_{\sigma \in \Sigma} g(\alpha_\sigma)$ between the sets $\{g(0)\}$ and $\{g(\alpha_\sigma) : \sigma \in \Sigma\} \cup \operatorname{acp}_{\sigma \in \Sigma} g(\alpha_\sigma)$.

The proof is obvious if $\gamma < 0$. Now, we assume that $\gamma > 0$.

[5] Note that, in the proofs of both the lemmas, we did not make use of the assumption that the functions under consideration are Darboux functions. We have thus proved the following considerably stronger property: Let $f : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ be a function such that, for an arbitrary $\alpha \in \mathbb{R}$, there exists $x_\alpha, y_\alpha \in \mathbb{R}$ such that $f|_{[x_\alpha, y_\alpha]}$ is a continuous function and $\alpha \in Int(f([x_\alpha, y_\alpha]))$. Then, for each transformation $g : \mathbb{R} \to Z$, if $g \circ f$ is a Darboux function, g is a Darboux function, too.

Let $\sigma_0 \in \Sigma$ and let n_{σ_0} be a positive integer such that $\frac{1}{n_{\sigma_0}} \leq \alpha_{\sigma_0}$. Then $t([n_{\sigma_0}, n_{\sigma_0} + \frac{1}{2}]) = [0, \frac{1}{n_{\sigma_0}}] \subset [0, \alpha_{\sigma_0}]$, and $A = (g \circ t)([n_{\sigma_0}, n_{\sigma_0} + \frac{1}{2}])$ is a connected set such that $g(0) \in A$ and $(\{g(\alpha_\sigma) : \sigma \in \Sigma\} \cup \mathrm{acp}_{\sigma \in \Sigma}\, g(\alpha_\sigma)) \cap A \neq \emptyset$. This means that $A \cap K \neq \emptyset$, and so, $g([0, \alpha_{\sigma_0}]) \cap K \neq \emptyset$. This finishes the proof of the fact that g is a Darboux function and, thereby, the proof of the condition $\hat{C} \neq \mathcal{F}_D$ is finished.

Now, we shall show that $\mathcal{F}_D \subset \mathcal{D}^*_Z$.

To prove the above inclusion, we shall demonstrate that if $\xi \notin D^*_Z$, then $\xi \notin \mathcal{F}_D$.

Let $\xi : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ be an arbitrary Darboux function such that there exists a point $\alpha^* \in \mathbb{R}$ of bilateral uniform discontinuity of f, i.e. there exists $\varepsilon^* > 0$ such that $\xi([x, x+\frac{1}{n}))\setminus(\alpha^*-\varepsilon^*, \alpha^*+\varepsilon^*) \neq \emptyset \neq \xi((x-\frac{1}{n}, x])\setminus(\alpha^*-\varepsilon^*, \alpha^*+\varepsilon^*)$ for any $x \in \xi^{-1}(\alpha^*)$ and $n \in \mathbb{N}$.

Now, we shall construct a function $g^* : \mathbb{R} \to Z$ which is not a Darboux function, such that $g^* \circ \xi$ is a Darboux function. Fix two distinct points $z_1, z_2 \in Z$. Let $g^*_1 : [\alpha^* - \varepsilon^*/2, \alpha^* + \varepsilon^*/2] \to Z$ be defined by the formula

$$g^*_1(x) = \begin{cases} z_1 & \text{if } x = \alpha^*, \\ z_2 & \text{if } x \neq \alpha^*. \end{cases}$$

Let $g^*_2 : (-\infty, \alpha^* - \varepsilon^*/2) \cup (\alpha^* + \varepsilon^*/2, +\infty)$ be a function mapping every interval onto (the whole space) Z. Then $g^* = g^*_1 \nabla g^*_2 : \mathbb{R} \xrightarrow{\text{onto}} Z$ is the sought-for function.

To finish this proof, it is enough to show that $\mathcal{F}_D \neq \mathcal{D}^*_Z$. It is easy to verify that the function $\varphi : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ defined by formula

$$\varphi(x) = \begin{cases} x & \text{for } x \leq 0, \\ \frac{1}{x}|\sin\frac{1}{x}| + \frac{1}{x} & \text{for } x > 0 \end{cases}$$

is a Darboux function belonging to $\mathcal{D}^*_Z$.

Let s_1, s_2 be two distinct points of Z. Then let $\zeta_1 : (-\infty, 1] \to \mathbb{R}$ be defined in the following way:

$$\zeta_1(x) = \begin{cases} s_1 & \text{if } x \leq 0, \\ s_2 & \text{if } x \in (0, 1]. \end{cases}$$

Let $\zeta_2 : (1, +\infty) \to Z$ be a function mapping every interval onto (the whole space) Z. Put $\zeta = \zeta_1 \nabla \zeta_2 : \mathbb{R} \to Z$. Of course, $\zeta \circ \varphi$ is a Darboux function, and ζ is not.

So, the proof is finished. □

Put $\mathcal{F}'_D = D \setminus \mathcal{F}_D$ where D is the set of all Darboux surjections $f : \mathbb{R} \to \mathbb{R}$. It is easy to show that there exists a Darboux surjection f such that the set of all discontinuity points of f is a singleton and $f \in \mathcal{F}'_D$. Of course, this function is quasi-continuous. It is interesting to consider the connection between the class of quasi-continuous Darboux surjections and the sets $\mathcal{F}_D$ and $\mathcal{F}'_D$. The following theorem show that $\mathcal{F}_D$ and $\mathcal{F}'_D$ are dense sets in the set of all quasi-continuous Darboux surjections.

Theorem 2 *Every quasi-continuous Darboux surjection $f : \mathbb{R} \to \mathbb{R}$ is the limit of uniformly convergent sequences $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}_D$ and $\{g_n\}_{n=1}^{\infty} \subset \mathcal{F}'_D$.*

PROOF. Let f be an arbitrary quasi-continuous Darboux function mapping the real line onto the real line. It is sufficient to show that:

(1) for each $n \in \mathbb{N}$, there exist $f_n \in \mathcal{F}_D$ and $g_n \in \mathcal{F}'_D$ such that

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \leq \frac{1}{n} \qquad \text{and} \qquad \sup_{x \in \mathbb{R}} |g_n(x) - f(x)| \leq \frac{1}{n}.$$

Now, let $n_0 \in \mathbb{N}$.

Denote $I_k = \left[\frac{k}{n_0}, \frac{k+1}{n_0}\right]$ for $k = 0, \pm 1, \pm 2, \ldots$. Then $\bigcup_{k=-\infty}^{+\infty} I_k = \mathbb{R}$.

As f is a Świątkowski function (see [HP]), for any $k = 0, \pm 1, \pm 2, \ldots$, there exists x_k such that f is continuous at x_k, and $f(x_k) \in \operatorname{Int} I_k$. Let $\delta_k > 0$ ($k = 0, \pm 1, \pm 2, \ldots$) be a number such that $f([x_k - \delta_k, x_k + \delta_k]) \subset \operatorname{Int} I_k$. Finally, for $k = 0, \pm 1, \pm 2, \ldots$, let $\xi_k : [x_k - \delta_k, x_k + \delta_k] \overset{\text{onto}}{\longrightarrow} I_k$ be a function linear on the segments $[x_k - \delta_k, x_k - \frac{1}{3}\delta_k]$, $[x_k - \frac{1}{3}\delta_k, x_k + \frac{1}{3}\delta_k]$, $[x_k + \frac{1}{3}\delta_k, x_k + \delta_k]$, such that $\xi_k(x_k - \delta_k) = f(x_k - \delta_k)$, $\xi_k(x_k + \delta_k) = f(x_k + \delta_k)$, $\xi_k(x_k - \frac{1}{3}\delta_k) = \frac{k+1}{n_0}$ and $\xi_k(x_k + \frac{1}{3}\delta_k) = \frac{k}{n_0}$. Then we put

$$f_{n_0} = \nabla_{k=-\infty}^{+\infty} \xi_k \ \nabla\ f|_{\mathbb{R} \setminus \bigcup_{k=-\infty}^{+\infty} [x_k - \delta_k, x_k + \delta_k]}.$$

It is not hard to verify that f_{n_0} is a Darboux function satisfying $\sup_{x \in \mathbb{R}} |f_{n_0}(x) - f(x)| \leq \frac{1}{n_0}$. According to Lemma 2, we deduce that $f_{n_0} \in \mathcal{F}_D$.

At present, we construct a function g_{n_0}. We consider the level $f^{-1}(0)$. Let $\{x_\alpha\}_{\alpha < \Omega}$ be a transfinite sequence consisting of all elements of $f^{-1}(0)$, where Ω is the smallest uncountable ordinal number (of course, this sequence need not consist of distinct elements).

Now, we consider x_0. If, for each $\delta > 0$, $f([x_0, x_0 + \delta)) \setminus \left[-\frac{1}{5n_0}, \frac{1}{5n_0}\right] \neq \emptyset$, then we put $A_0^1 = \emptyset$. In the opposite case, let δ_0^1 be a positive real number such that $f([x_0, x_0 + \delta_0^1]) \subset \left[-\frac{1}{5n_0}, \frac{1}{5n_0}\right]$ and we put $A_0^1 = (x_0, x_0 + \delta_0^1)$. The analogous construction is made on the left-hand side of the point x_0, which leads to the definition of the set A_0^2. Let $A_0 = A_0^1 \cup A_0^2$.

Suppose that we have constructed the set A_α for each x_α where $\alpha < \beta < \Omega$. If $x_\beta \in \bigcup_{\alpha<\beta} A_\alpha$, then we put $A_\beta = \emptyset$. In the opposite case, similarly as for x_0, we define the set A_β for x_β.

Denote $A = \bigcup_{\alpha<\Omega} A_\alpha$. If $A = \emptyset$, then let $g_{n_0} = f$. If $A \neq \emptyset$, then we define in the set A the equivalence relation $*$ in the following way:

$$p * r \qquad \text{if and only if} \qquad p - r \qquad \text{is a rational number.}$$

By the letter B we denote the set of all equivalence classes of the relation $*$. Let h be a surjection B onto $\left[-\frac{1}{3n_0}, \frac{1}{3n_0}\right]$. Then we define $g_{n_0} : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ as follows:

$$g_{n_0}(x) = \begin{cases} h([x]_*) & \text{if } x \in A, \\ f(x) & \text{if } x \notin A, \end{cases}$$

where $[x]_*$ denotes the equivalence class containing x. Note that if J is an arbitrary closed interval, then

$$g_{n_0}(J) = \begin{cases} f(J) & \text{if } A \cap J = \emptyset, \\ f(J) \cup \left[-\frac{1}{3n_0}, \frac{1}{3n_0}\right] & \text{if } A \cap J \neq \emptyset. \end{cases}$$

This means that g_{n_0} is a Darboux function. It is easy to see that $\sup_{x\in\mathbb{R}} |g_{n_0}(x) - f(x)| \leq \frac{2}{3n_0} < \frac{1}{n_0}$.

Now, we define the function $g^*_{n_0}$ in the following way: $g^*_{n_0}(0) = 0$; $g^*_{n_0}(x) = 1$ for $x \in \left[-\frac{1}{5n_0}, \frac{1}{5n_0}\right]\setminus\{0\}$, with $g^*_{n_0}$ mapping each interval contained in $\left(-\infty, -\frac{1}{5n_0}\right) \cup \left(\frac{1}{5n_0}, +\infty\right)$ onto (the whole) real line. Then $g^*_{n_0} \circ g_{n_0}$ is a Darboux function, but $g^*_{n_0}$ is not, which proves that $g_{n_0} \in \mathcal{F}'_D$.

The proof of condition (1) is completed. □

The theorem we have proved "provokes" one to formulate the following open problem: What other classes of functions (except quasi-continuous Darboux functions) possess the property that each function from this class is the limit of uniformly convergent sequences of functions from both the classes $\mathcal{F}_D$ and $\mathcal{F}'_D$?

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