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FINITE ADDITIVITY AND CLOSEST APPROXIMATIONS

Abstract

Suppose that Ω is a set, F is a field of subsets of Ω, and $A(\mathbb{R})(F)$ is the set of all real-valued finitely additive functions defined on F. For each $\xi \in A(\mathbb{R})(F)$ let $A(\xi) = \{\eta : \eta \in A(\mathbb{R})(F),\ \xi - \eta \text{ bounded}\}$. It is shown that for each ξ in $A(\mathbb{R})(F)$, for certain subsets of $A(\xi)$ with special closure properties, theorems about closest approximations, functional equations and decompositions hold.

1 Introduction

In a large variety of spaces, near - point, or closest approximation considerations frequently arise, either explicitly, in the form of theorems, or implicitly, as parts of various arguments. Among these spaces, various function spaces play a prominent role (see [1,2,3,4,6,7]), and it is for such a function space, specifically a space of finitely additive set functions, not necessarily bounded, that we carry out our investigations concerning closest approximations in this paper.

Suppose that Ω is a set, F is a field of subsets of Ω, $A(\mathbb{R})(F)$ is the set of all real-valued functions with domain F that are finitely additive, $AB(\mathbb{R})(F)$ is the set of all bounded elements of $A(\mathbb{R})(F)$ and $AB(\mathbb{R})(F)^+$ is the set of all nonnegative - valued elements of $A(\mathbb{R})(F)$. In this paper, all integrals will be refinement - wise limits of the appropriate sums (see section 2).

Many subsets of $AB(\mathbb{R})(F)$ have a near point property, and we list some of these below.

Suppose that μ is in $AB(\mathbb{R})(F)^+$.

1) Let $A_\mu = \{\xi : \xi \text{ in } AB(\mathbb{R})(F), \xi \text{ absolutely continuous with respect to } \mu\}$ (ϵ,δ definition).

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2) Let C_μ denote the set to which ξ belongs iff ξ is in $AB(\mathbb{R})(F)$ and is "μ - refinement - continuous", i. e., if $0 < c$, then there is $d > 0$ and subdivision D of Ω such that if I is in F and $I \subseteq V$ for some V in D and $\mu(I) < d$, then $|\xi(I)| < c$.

3) Suppose that α is a function from F into $\exp(\mathbb{R})$ with bounded range union. Let $\mathcal{I}_\alpha$ denote $\{\xi : \xi \text{ in } AB(\mathbb{R})(F),\ \int_\Omega \alpha\xi \text{ exists }\}$.

4) For some $K \geq 0$ let

$$\mathrm{Lip}\,(\mu, K) = \{\xi : \xi \text{ in } AB(\mathbb{R})(F),\ K\mu - \int |\xi| \text{ in } AB(\mathbb{R})(F)^+\}$$

.

Now, if P is any of the above subsets and ξ is in $AB(\mathbb{R})(F)$, then there is an element ν of P such that if γ is in P and $\gamma \neq \nu$, then

$$\int_\Omega |\xi(I) - \nu(I)| < \int_\Omega |\xi(I) - \gamma(I)|.$$

This fact is a special case of the following closest approximation theorem [1]:

Theorem 1.1 *Suppose that P is a subset of $AB(\mathbb{R})(F)$ satisfying the following conditions:*

1) If η is in P, ζ is in $AB(\mathbb{R})(F)$ and $\int|\eta| - \int|\zeta|$ is in $AB(\mathbb{R})(F)^+$, then ζ is in P, and

2) If μ is in $AB(\mathbb{R})(F)^+$ and λ is the function with domain F given by

$$\lambda(V) = \sup\{\eta(V) : \eta \text{ in } P \cap AB(\mathbb{R})(F)^+, \mu - \eta \text{ in } AB(\mathbb{R})(F)^+\},$$

then λ is in $P \cap AB(\mathbb{R})(F)^+$.

Then, for each ξ in $AB(\mathbb{R})(F)$ there is an element ν of P such that if γ is in P and $\gamma \neq \nu$, then $\int_\Omega |\xi(I) - \nu(I)| < \int_\Omega |\xi(I) - \gamma(I)|$.

Now, for the above setting, it can be easily shown that for ξ and ν as described, γ in P and V in F, that $\int_V |\xi(I) - \nu(I)| \leq \int_V |\xi(I) - \gamma(I)|$. For a subset of $AB(\mathbb{R})(F)$ for which the uniqueness of minimizing functions is not necessarily true, the immediately preceding inequality is the best possible. It is this sort of inequality that we obtain as a generalization of Theorem 1.1 and which we state below immediately after Definitions 1.1 and 1.2.

Definition 1.1 *If η is in $A(\mathbb{R})(F)$ and V is in F, then $\eta^{[V]}$ is the function from F into $\mathbb{R}$ given by $\eta^{[V]}(I) = \eta(V \cap I)$; we take the usual liberties with set intersections; clearly $\eta^{[V]}$ is in $A(\mathbb{R})(F)$.*

Definition 1.2 *If ξ is in $A(\mathbb{R})(F)$, then*

$$A(\xi) = \{\eta : \eta \text{ in } A(\mathbb{R})(F), \xi - \eta \text{ in } AB(\mathbb{R})(F)\}.$$

Clearly $\{(\zeta, \nu) : \zeta - \nu$ in $AB(\mathbb{R})(F)\}$ is an equivalence relation, so that if each of η and ζ is in $A(\mathbb{R})(F)$, then $A(\zeta) = A(\eta)$ iff $A(\zeta)$ and $A(\eta)$ have an element in common.

Theorem 3.1 *Suppose that ξ is in $A(\mathbb{R})(F)$, $M \subseteq A(\xi)$ and the following three statements are true:*

1) *If each of η and ζ is in M, then so is each of $\int max\{\eta, \zeta\}$ and $\int min\{\eta, \zeta\}$ (see section 2 concerning the existence of certain integrals).*

2) *If V is in F, $V \neq \Omega$ and each of η and ζ is in M, then $\eta^{[V]} + \zeta^{[\Omega - V]}$ is in M.*

3) *Suppose that λ is a function from F into $\mathbb{R}$ and $\{\mu_k\}_{k=1}^{\infty}$ is a sequence of elements of M such that either:*

   i) *for each V in F, $\mu_n(V) \leq \mu_{n+1}(V)$ for all n, and*

   $$\lambda(V) = sup\{\mu_n(V) : n \text{ a positive integer}\},$$

   or

   ii) *for each V in F, $\mu_n(V) \geq \mu_{n+1}(V)$ for all n, and*

   $$\lambda(V) = inf\{\mu_n(V) : n \text{ a positive integer}\}.$$

   Then λ is in M.

Then there is λ' in M such that if η is in M and V is in F, then

$$\int_V |\xi(I) - \lambda'(I)| \leq \int_V |\xi(I) - \eta(I)|.$$

As was remarked above, the inequality of the conclusion of Theorem 3.1 is the "best possible". To see this, let $F_{(0;1]}$ denote the collection of all unions of finite subcollections of $\{(p; q] : 0 \leq p \leq q \leq 1\}$. As is well-known, $F_{(0;1]}$ is a field of subsets of $(0; 1]$. By well-known methods there is an element μ in $AB(\mathbb{R})(F_{(0;1]})^+$ such that $\{\xi : \xi$ in $AB(\mathbb{R})(F_{(0;1]}), \int |\xi| = \mu\}$, denoted by $V(\mu)$, contains more than one element. Letting 0 denote the zero function on $F_{(0;1]}$, we see that $(0, V(\mu))$ satisfies the hypothesis of Theorem 3.1 with the further property that if ρ_1 and ρ_2 are in $V(\mu)$ and V is in $F_{(0;1]}$, then

$$\int_V |0(I) - \rho_1(I)| = \int_V |0(I) - \rho_2(I)|.$$

As a further example of a subset of $A(\mathbb{R})(F)$ that satisfies the hypothesis of Theorem 3.1, but not necessarily the hypothesis nor the conclusion of Theorem 1.1, we consider the following generalization of the example of the immediately preceding paragraph.

Example: Suppose that H is a closed and bounded subset of $\mathbb{R}$, ξ is in $AB(\mathbb{R})(F)$, and

$$M = \left\{ \int \alpha\xi : \alpha \text{ a function from } F \text{ into } H, \int_\Omega \alpha\xi \text{ exists} \right\}.$$

With reference to establishing the conditions of the hypothesis of Theorem 3.1 for the above collection, we refer the reader to [2] as well as section 2 of this paper for the differential equivalence (see also [5]) consequences and closure properties that imply them.

We now continue our discussion of the theorems of this paper.

We show the following functional equation theorem (see section 4):

Theorem 4.1 *Suppose that each of ξ_1 and ξ_2 is in $A(\mathbb{R})(F)$ and $M \subseteq A(\xi_i)$ for $i = 1, 2$, so that $A(\xi_1) = A(\xi_2)$. Suppose that M satisfies the hypothesis of Theorem 3.1. Suppose that if $i = 1, 2$, then λ_i is an element of M such that for each V in F,*

$$\int_V |\xi_i(I) - \lambda_i(I)| = \inf\left\{ \int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M \right\}.$$

It follows that if Q is max or min, then $\int_\Omega Q\{\xi_1, \xi_2\}$ exists, $M \subseteq A(\int Q\{\xi_1, \xi_2\})$ (see Theorem 2.2) and for each V in F,

$$\int_V \left| \int_I Q\{\xi_1(J), \xi_2(J)\} - \int_I Q\{\lambda_1(J), \lambda_2(J)\} \right| =$$

$$\inf\left\{ \int_V \left| \int_I Q\{\xi_1(J), \xi_2(J)\} - \eta(I) \right| : \eta \text{ in } M \right\}.$$

We conclude the paper with a converse to Theorem 4.1 in the form of the following decomposition theorem:

Theorem 5.1 *Suppose that each of ξ_1 and ξ_2 is in $A(\mathbb{R})(F)$ and $M \subseteq A(\xi_i)$ for $i = 1, 2.$, so that $A(\xi_1) = A(\xi_2)$. Suppose that if ξ is ξ_1 or ξ_2, then (ξ, M) satisfies the hypothesis of Theorem 3.1. Suppose that ρ is in M, Q is max or min and for each V in F,*

$$\int_V \left| \int_I Q\{\xi_1(J), \xi_2(J)\} - \rho(I) \right| = \inf\left\{ \int_V \left| \int_I Q\{\xi_1(J), \xi_2(J)\} - \eta(I) \right| : \eta \in M \right\}.$$

Then for $i = 1, 2$ there is μ_i in M such that *i)* $\rho = \int Q\{\mu_1, \mu_2\}$ *and*

ii) for $i = 1, 2$ *and each* V *in* F,

$$\int_V |\xi_i(I) - \mu_i(I)| = \inf\left\{\int_V |\xi_i(I) - \eta(I)| : \eta \in M\right\}.$$

2 Preliminary Theorems and Definitions

If V is in F, then the statement that D is a subdivision of V means that D is a finite collection of mutually disjoint sets of F with union V. The statement that E is a refinement of H, denoted by "$E << H$"; means that for some V in F, each of E and H is a subdivision of V and each element of E is a subset of some element of H.

We shall let $r(F)$ denote the set of all functions from F into $\exp(\mathbb{R})$. If α is in $r(F)$, V is in F and $E << \{V\}$, then the statement that b is an α - function on E means that b is a function with domain E such that for each I in E, $b(I)$ is in $\alpha(I)$.

As stated in the introduction, in this paper all integrals shall be refinement-wise limits of the appropriate sums; thus, if α is in $r(F)$ and V is in F, then the statement that K is an integral of α on V means that K is in $\mathbb{R}$ and if $0 < c$, then there is $D << \{V\}$ such that if $E << D$ and b is an α - function on E, then $|K - \sum_E b(I)| < c$; K is unique and is denoted by $\int_V \alpha(I)$, $\int_V \alpha(J)$, $\int_V \alpha$, etc. We refer the reader to [2] for the basic properties of these set function integrals, as well Kolmogoroff's [5] notion of differential equivalence and some of its immediate consequences, which shall be used in this paper, and of which we shall give a brief listing following Definition 2.1 and Convention 2.1 below.

We digress briefly for the following definition and convention.

Definition 2.1 *Suppose that S is a set, each of f and g is a function from S into a collection of sets, and w is a function such that*

$$\textit{(Range union)}(f) \times \textit{(Range union)}(g) \subseteq \textit{Dom}(w).$$

Then

$$w(f, g) = \{(t, w(x, y)) : t \text{ in } S,\ (x, y) \text{ in } f(t) \times g(t)\}.$$

Convention 2.1 *If S is a set and h is a function from S into $\mathbb{R}$, then, for certain purposes in this paper, we shall regard h as "equivalent" to $\{(t, \{h(t)\}) : t \text{ in } S\}$; we shall, at times, combine "set-valued" and "single-valued" functions in accordance with this convention and above definition.*

For $D << \{\Omega\}$, we let $r(D)(\Sigma)(F)$ denote the set to which α belongs iff α is in $r(F)$ and $\{\sum_E b(I) : E << D,\ b \text{ an } \alpha - \text{function on } E\}$ is bounded.

For each $D << \{\Omega\}$ and β in $r(D)(\Sigma)(F)$ we let $L_D(\beta)$ and $G_D(\beta)$ denote, respectively, the function with domain F such that for each V in F, $L_D(\beta)$ and $G_D(\beta)$ are the sup and inf, respectively, of the (consequently) bounded set

$$\left\{\sum_E b(I) : E \text{ a subset of a refinement of } D, E << \{V\},\right.$$
$$\left. b \text{ a } \beta-\text{function on } E\right\}.$$

Theorem 2.1 *Suppose that $D << \{\Omega\}$, and suppose that each of α, β and γ is in $r(D)(\Sigma)(F)$. Then the following statements are true.*

1) If V is in F, $P << \{V\}$, $Q << \{V\}$ and H is a refinement of each of P and Q, then

$$\sum_P G_D(\alpha)(I) \le \sum_H G_D(\alpha)(J) \le \sum_H L_D(\alpha)(J) \le \sum_Q L_D(\alpha)(I),$$

so that we have the following existence and inequality:

$$\int_V G_D(\alpha)(I) \le \int_V L_D(\alpha)(I),$$

equality holding iff $\int_V \alpha(I)$ exists, in which case $\int_V G_D(\alpha)(I) = \int_V \alpha(I) = \int_V L_D(\alpha)(I)$.

2) If $\int_\Omega \alpha$ exists, then, for each V in F $\int_V \alpha$ exists and $\int \alpha$, which denotes $\{(V, \int_V \alpha) : V$ in $F\}$ is in $A(\mathbb{R})(F)$.

3) If $\int_\Omega \alpha$ exists, then $\int_\Omega |\alpha(I) - \int_I \alpha| = 0$, i. e., if $0 < c$, then there is $D << \{\Omega\}$ such that if $E << D$ and a is an α-function on E, then $\sum_E |a(I) - \int_I \alpha| < c$.

4) If ν is $r(F)$ and has bounded range union, then $\int_\Omega |\nu(I)||\alpha(I) - \int_I \alpha| = 0$, so that if V is in F, then $\int_V \nu(I)\alpha(I)$ exists iff $\int_V \nu(I) \int_I \alpha$ exists, in which case equality holds.

5) If Q is max or min and each of $\int_\Omega \beta$ and $\int_\Omega \gamma$ exists, then

$$\int_\Omega |Q\{\beta(I), \gamma(I)\} - Q\{\int_I \beta, \int_I \gamma\}| = 0,$$

so that if V is in F, then $\int_V Q\{\beta(I), \gamma(I)\}$ exists iff $\int_V Q\{\int_I \beta, \int_I \gamma\}$ exists, in which case equality holds.

We now discuss some inequalities and their integral existence implications. If S is a finite set and each of f and g is a function from S into $\mathbb{R}$, then

$$\sum_S \min\{f(x), g(x)\} \le \min\left\{\sum_S f(x), \sum_S g(x)\right\}$$

$$\leq \max\left\{\sum_S f(x), \sum_S g(x)\right\} \leq \sum_S \max\{f(x), g(x)\}.$$

This implies that if V is in F, $D << \{V\}$, $E << D$, $H << D$ and each of η and ζ is in $A(\mathbb{R})(F)$, then

$$\sum_E \min\{\eta(J), \zeta(J)\} \leq \sum_D \min\{\eta(I), \zeta(I)\}$$

$$\leq \sum_D \max\{\eta(I), \zeta(I)\} \leq \sum_H \max\{\eta(J), \zeta(J)\},$$

so that $\int_V \min\{\eta(J), \zeta(J)\}$ exists iff $\{\sum_P\{min\{\eta(J), \zeta(J)\} : P << \{V\}\}$ is bounded below, and $\int_V \max\{\eta(J), \zeta(J)\}$ exists iff $\{\sum_P \max\{\eta(J), \zeta(J)\} : P << \{V\}\}$ is bounded above.

We now prove an integral existence theorem for elements of $A(\mathbb{R})(F)$.

Theorem 2.2 *If each of η and ζ is in $A(\mathbb{R})(F)$, then the following three statements are equivalent:*

1) There is ξ in $A(\mathbb{R})(F)$ such that each of η and ζ is in $A(\xi)$,

2) If V is in F, then $\int_V max\{\eta(I), \zeta(I)\}$ exists, and

3) If V is in F, then $\int_V min\{\eta(I), \zeta(I)\}$ exists.

Furthermore, in case 1) each of $\int max\{\eta, \zeta\}$ and $\int min\{\eta, \zeta\}$ is in $A(\xi)$.

PROOF. Suppose that 1) is true and Q is max or min. Suppose that V is in F. If $E << \{V\}$, then

$$\left|\sum_E Q\{\eta(I), \zeta(I)\}\right| = \left|\left[\sum_E Q\{\eta(I) - \xi(I), \zeta(I) - \xi(I)\}\right] + \xi(V)\right|$$

$$\leq \sum_E |Q\{\eta(I) - \xi(I), \zeta(I) - \xi(I)\}| + |\xi(V)|$$

$$\leq \left(\sum_E |\eta(I) - \xi(I)|\right) + \left(\sum_E |\zeta(I) - \xi(I)|\right) + |\xi(V)|$$

$$\leq \left(\int_V |\eta(I) - \xi(I)|\right) + \left(\int_V |\zeta(I) - \xi(I)|\right) + |\xi(V)|.$$

Therefore $\int_V Q\{\eta(I), \zeta(I)\}$ exists. Therefore 1) implies each of 2) and 3).

Suppose that 2) or 3) is true. Let $Q = \max$ or min and $\xi = \int Q\{\eta, \zeta\}$. We immediately see that each of $\xi - \eta$ and $\xi - \zeta$ is in $AB(\mathbb{R})(F)$, so that each of η and ζ is in $A(\xi)$. Therefore each of 2) and 3) implies 1).

Therefore 1), 2) and 3) are equivalent.

Again, assume that 1) is true and $Q = \max$ or min. Suppose that $E << \{\Omega\}$. Then

$$\sum_E \left| \left(\int_I Q\{\eta(J), \zeta(J)\} \right) - \xi(I) \right| = \sum_E \left| \int_I Q\{\eta(J) - \xi(J), \zeta(J) - \xi(J)\} \right|$$
$$\leq \sum \left(\int_I |\eta(J) - \xi(J)| + \int_I |\zeta(J) - \xi(J)| \right)$$
$$= \int_\Omega |\eta(J) - \xi(J)| + \int_\Omega |\zeta(J) - \xi(J)|.$$

Therefore $\int Q\{\eta, \zeta\}$ is in $A(\xi)$. □

We now prove a theorem involving upper and lower integrals and the operations max and min.

Theorem 2.3 *Suppose that $D << \{\Omega\}$ and each of α, β, $min\{\alpha, \beta\}$ and $max\{\alpha, \beta\}$ is in $r(D)(\Sigma)(F)$. Then, if V is in F and (P, Q) is either (G_D, min) or (L_D, max), then the following existence and equality holds:*

$$\int_V P(Q\{\alpha, \beta\})(I) = \int_V Q\{P(\alpha)(I), P(\beta)(I)\}.$$

PROOF. W.l.o.g. let $(P, Q) = (L_D, \max) = (L, \max)$. Suppose that W is in F. First, if $E << \{W\}$ and E is a subset of some refinement of D, then

$$\sum_E \max\{L(\alpha)(I), L(\beta)(I)\} \leq \sum_E L(\max\{\alpha, \beta\})(I) \leq L(\max\{\alpha, \beta\})(W),$$

so that

$$L(\max\{L(\alpha), L(\beta)\})(W) \leq L(\max\{\alpha, \beta\})(W) \leq L(\max\{L(\alpha), L(\beta)\})(W),$$

so that

$$L(\max\{L(\alpha), L(\beta)\})(W) = L(\max\{\alpha, \beta\})(W). \tag{2.1}$$

Secondly, if V is in F and $E << \{V\}$, then

$$\sum_E \max\left\{ \int_I L(\alpha)(J), \int_I L(\beta)(J) \right\} \leq \sum_E \int_I L(\max\{\alpha, \beta\})(J)$$
$$= \int_V L(\max\{\alpha, \beta\})(J),$$

so that from this, differential equivalence and (2.1), we have, respectively, the following existence and equality:

$$\begin{aligned}\int_V \max\{L(\alpha)(I), L(\beta)(I)\} &= \int_V L(\max\{L(\alpha), L(\beta)\})(I)\\ &= \int_V L(\max\{\alpha,\beta\})(J).\end{aligned}$$

Therefore $\int_V L(\max\{\alpha,\beta\})(J) = \int_V \max\{L(\alpha)(I), L(\beta)(I)\}$.

The argument for G_D and min follows in a similar fashion.

3 A Closest Approximation Theorem for Subsets of $A(\mathbb{R})(F)$.

Lemma 3.1 *Suppose that ξ is in $A(\mathbb{R})(F)$, $M \subseteq A(\xi)$ and if V is in F and is not Ω and each of η and ζ is in M, then $\eta^{[V]}+\zeta^{[\Omega-V]}$ is in M. Suppose that ω is the function with domain F given by $\omega(V) = inf\{\int_V |\xi(I)-\eta(I)| : \eta$ in $M\}$. Then ω is in $AB(\mathbb{R})(F)^+$.*

PROOF. Obviously ω is nonnegative-valued.

Now suppose that V and W are mutually disjoint sets of F. If η is in M, then

$$\omega(V)+\omega(W) \le \int_V |\xi(I)-\eta(I)| + \int_W |\xi(I)-\eta(I)| = \int_{V\cup W} |\xi(I)-\eta(I)|.$$

Therefore $\omega(V)+\omega(W) \le \omega(V\cup W)$.

Suppose that $0 < c$. There is η and ζ, each in M, such that $\int_V |\xi(I) - \eta(I)| < \omega(V) + c/2$ and $\int_W |\xi(I) - \zeta(I)| < \omega(W) + c/2$, so that, if $\lambda = \eta^{[V]} + \zeta^{[\Omega-V]}$, then, since λ is in M and $W \subseteq \Omega - V$, it follows that

$$\begin{aligned}\omega(V\cup W) &\le \int_{V\cup W} |\xi(I)-\lambda(I)| = \int_V |\xi(I)-\lambda(I)| + \int_W |\xi(I)-\lambda(I)|\\ &= \int_V |\xi(I)-\eta(I)| + \int_W |\xi(I)-\zeta(I)| < \omega(V)+c/2+\omega(W)+c/2,\end{aligned}$$

so that $\omega(V\cup W) < \omega(V)+\omega(W)+c$. Therefore $\omega(V)+\omega(W) \le \omega(V\cup W) \le \omega(V)+\omega(W)$, so that $\omega(V\cup W) = \omega(V)+\omega(W)$. Therefore ω is in $AB(\mathbb{R})(F)^+$.

PROOF OF THEOREM 3.1 Let ω be the function with domain F given by

$$\omega(V) = \inf\left\{\int_V |\xi(I)-\eta(I)| : \eta \text{ in } M\right\}.$$

By Lemma 3.1 it follows that ω is in $AB(\mathbb{R})(F)^+$, so that if η is in M, then $\int |\xi - \eta| - \omega$ is in $AB(\mathbb{R})(F)^+$.

There is a sequence $\{\eta_k\}_{k=1}^{\infty}$ of elements of M such that for each n,

$$\left(\int_{\Omega} |\xi(I) - \eta_n(I)|\right) - \omega(\Omega) \leq 2^{-(n+1)},$$

so that if V is in F, then $\left(\int_V |\xi(I) - \eta_n(I)|\right) - \omega(V) \leq 2^{-(n+1)}$. For each n there is, by hypothesis and induction, a sequence $\{\mu_n^k\}_{k=n}^{\infty}$ of elements of M such that $\mu_n^n = \eta_n$ and, if $n \leq m$, then $\mu_n^{m+1} = \int \max\{\eta_{m+1}, \mu_n^m\}$. It is obvious that for each positive integer n and m with $m \geq n$, $\mu_n^{m+1} - \mu_n^m$ is in $AB(\mathbb{R})(F)^+$. It also follows routinely that for each positive integer n and m with $m \geq n+1$, $\mu_n^m - \mu_{n+1}^m$ is in $AB(\mathbb{R})(F)^+$.

Suppose that n is a positive integer. For each V in F,

$$\left(\int_V |\xi(I) - \mu_n^n(I)|\right) - \omega(V) \leq 2^{-(n+1)}.$$

Suppose that h is a positive integer$\geq n$ such that for each X in F,

$$\left(\int_X |\xi(I) - \mu_n^h(I)|\right) - \omega(X) \leq \sum_{k=n}^{h} 2^{-(k+1)}.$$

Suppose that V is in F. Then

$$\int_V |\xi(I) - \mu_n^{h+1}(I)| = \int_V |\xi(I) - \max\{\eta_{h+1}(I), \mu_n^h(I)\}|.$$

Suppose that $D << \{V\}$. Let $D' = \{I : I \text{ in } D, \eta_{h+1}(I) < \mu_n^h(I)\}$ and $P' = \cup_{D'} I$. Then

$$\begin{aligned}
\sum_D |\xi(I) - \max\{\eta_{h+1}(I), \mu_n^h(I)\}| &= \sum_{D'} |\xi(I) - \mu_n^h(I)| + \sum_{D-D'} |\xi(I) - \eta_{h+1}(I)| \\
&\leq \sum_{D'} \int_I |\xi(J) - \mu_n^h(J)| + \sum_{D-D'} \int_I |\xi(J) - \eta_{h+1}(J)| \\
&= \int_{P'} |\xi(J) - \mu_n^h(J)| + \int_{V-P'} |\xi(J) - \eta_{h+1}(J)| \\
\leq \left(\sum_{k=n}^{h} 2^{-(k+1)}\right) + \omega(P') + 2^{-(h+2)} + \omega(V - P') &= \left(\sum_{k=n}^{h+1} 2^{-(k+1)}\right) + \omega(V).
\end{aligned}$$

It therefore follows that $\int_V |\xi(I) - \mu_n^{h+1}(I)| \leq (\sum_{k=n}^{h+1} 2^{-(k+1)}) + \omega(V)$. Therefore, if m is a positive integer $\geq n$ and V is in F, then $\int_V |\xi(I) - \mu_n^m(I)| \leq (\sum_{k=n}^{m} 2^{-(k+1)}) + \omega(V)$.

Suppose that n is a positive integer. We first show that if V is in F and m is a positive integer $\geq n$, then $\mu_n^m(V) \leq \mu_n^{m+1}(V) \leq 2^{-n} + \omega(V) + \xi(V)$. The first portion of the inequality has already been established. Again, suppose that m is a positive integer $\geq n$ and V is in F. Then

$$\mu_n^m(V) = \mu_n^m(V) - \xi(V) + \xi(V) \leq \int_V |\xi(I) - \mu_n^m(I)| + \xi(V)$$
$$\leq \left(\sum_{k=n}^{m} 2^{-(k+1)}\right) + \omega(V) + \xi(V) \leq 2^{-n} + \omega(V) + \xi(V).$$

Let λ_n be the function with domain F given by

$$\lambda_n(V) = \sup\{\mu_n^m(V) : m \text{ a positive integer } \geq n\}.$$

By hypothesis, λ_n is in M. Furthermore, for each μ_n^m, $\lambda_n - \mu_n^m$ is in $AB(\mathbb{R})(F)^+$, so that for each V in F, $\int_V |\lambda_n(I) - \mu_n^m(I)| = \lambda_n(V) - \mu_n^m(V)$, so that

$$\int_V |\xi(I) - \lambda_n(I)| \leq \int_V |\xi(I) - \mu_n^m(I)| + \int_V |\mu_n^m(I) - \lambda_n(I)|$$
$$\leq \left(\sum_{k=n}^{m} 2^{-(k+1)}\right) + \omega(V) + \lambda_n(V) - \mu_n^m(V) \to 2^{-n} + \omega(V) + 0 \text{ as } m \to \infty,$$

so that $\int_V |\xi(I) - \lambda_n(I)| \leq 2^{-n} + \omega(V)$.

We see that if V is in F and n is a positive integer, then $\lambda_n(V) \geq \lambda_{n+1}(V)$ and

$$\lambda_n(V) = \lambda_n(V) - \xi(V) + \xi(V) \geq -\int_V |\lambda_n(I) - \xi(I)| + \xi(V)$$
$$\geq -2^{-n} - \omega(V) + \xi(V) \geq -1 - \omega(V) + \xi(V).$$

Thus, let λ' denote the function with domain F given by

$$\lambda'(V) = \inf\{\lambda_n(V) : n \text{ a positive integer}\}.$$

By hypothesis, λ' is in M. Furthermore, for each n, $\lambda_n - \lambda'$ is in $AB(\mathbb{R})(F)^+$, and for each V in F, $\int_V |\lambda'(I) - \lambda_n(I)| = \lambda_n(V) - \lambda'(V)$, so that

$$\int_V |\xi(I) - \lambda'(I)| \leq \int_V |\xi(I) - \lambda_n(I)| + \int_V |\lambda_n(I) - \lambda'(I)|$$
$$\leq 2^{-n} + \omega(V) + \lambda_n(V) - \lambda'(V) \to 0 + \omega(V) + 0 \text{ as } n \to \infty,$$

so that $\omega(V) \leq \int_V |\xi(I) - \lambda'(I)| \leq \omega(V)$,so that $\int_V |\xi(I) - \lambda'(I)| = \omega(V)$. $\square$

Corollary 3.1 *Assume the hypothesis of Theorem 3.1. Suppose further that if $0 < c$, then there is η in M such that $\int_\Omega |\xi(I) - \eta(I)| < c$. Then ξ is in M.*

PROOF. By Theorem 3.1 there is λ in M such that if V is in F, then

$$\int_V |\xi(I) - \lambda(I)| = \inf\left\{\int_V |\xi(I) - \eta(I)| : \eta \text{ in } M\right\},$$

which, from the hypothesis is clearly 0. Therefore $\int_\Omega |\xi(I) - \lambda(I)| = 0$, so that ξ is λ, so that ξ is in M.

4 A Functional Equation

In this section we prove Theorem 4.1, as stated in the introduction. We begin with three lemmas.

The author wishes to thank the referees for their helpful suggestions concerning this paper, and one of the referees in particular concerning the following lemma.

Lemma 4.1 *Suppose that $0 \le K$, each of p, q, r and s is in $\mathbb{R}$, $p \le q$ and $|p - r| \le |p - s| + K$. Then $|q - max\{r, s\}| \le |q - s| + K$.*

PROOF. If $r \le s$, then the conclusion follows immediately. So suppose that $s < r$. We then wish to show that $|q - r| \le |q - s| + K$, or equivalently, $|q-r| - |q-s| \le K$. If $r \le q$, then, clearly $|q-r| = q-r < q-s \le |q-s| + K$. So next suppose that $s \le q < r$. If $p < s \le q < r$, then $r - s = r - p - (s - p) = |r - p| - |s - p| \le K$, so that $|q - r| - |q - s| \le |q - r - (q - s)| = r - s \le K$. If $s \le p \le q < r$, then $|q - r| \le |p - r| \le |p - s| + K \le |q - s| + K$. Finally, suppose that $q < s < r$. Again, since $p \le q < s < r$,

$$r - s = r - p - (s - p) = |p - r| - |p - s| \le K,$$

so that $|q - r| - |q - s| = r - q - (s - q) = r - s \le K$. Therefore the inequality is true. □

We now state an easy consequence of Lemma 4.1, giving a few remarks concerning justification.

Lemma 4.2 *If each of a, b, c and d is in $\mathbb{R}$ and $|a - c| \le |a - d| + K$ and $|b - d| \le |b - c| + K$, then*

$$|max\{a, b\} - max\{c, d\}| \le \begin{cases} |a - c| + K, & \text{if } b \le a \\ |b - d| + K, & \text{if } a \le b. \end{cases} \tag{4.1}$$

and

$$(4.2) \qquad |min\{a,b\} - min\{c,d\}| \leq \begin{cases} |a-c| + K, & \text{if } a \leq b \\ |b-d| + K, & \text{if } b \leq a. \end{cases}$$

Remark The proof of inequalities (4.1) is a matter of routine substitution. Then one may use these inequalities in conjunction with the equation $\min\{x,y\} = -\max\{-x,-y\}$ to establish inequalities (4.2); we leave the details to the reader.

Lemma 4.3 *Assume the hypothesis of Theorem 4.1. Then there is a real nonnegative-valued function K with domain F such that*

1) $\int_\Omega K(I) = 0$ *and*

2) If V is in F, $i = 1,2$, $j = 1,2$, and $i \neq j$, then

$$|\xi_i(V) - \lambda_i(V)| \leq |\xi_i(V) - \lambda_j(V)| + K(V).$$

INDICATION OF PROOF. Suppose that V is in F, $i = 1,2$, $j = 1,2$, and $i \neq j$. Then

$$\begin{aligned} |\xi_i(V) - \lambda_i(V)| &= [|\xi_i(V) - \lambda_i(V)| - \int_V |\xi_i(I) - \lambda_i(I)|]_V^{1,i} + \int_V |\xi_i(I) - \lambda_i(I)| \\ &\leq |[\quad]_V^{1,i}| + \int_V |\xi_i(I) - \lambda_j(I)| \\ &= |[\quad]_V^{1,i}| + \left[(\int_V |\xi_i(I) - \lambda_j(I)|) - |\xi_i(V) - \lambda_j(V)| \right]_V^{2,i} + |\xi_i(V) - \lambda_j(V)|. \end{aligned}$$

Let $K_i(V) = |[\quad]_V^{1,i}| + [\quad]_V^{2,i}$. Then $\int_\Omega K_i(V) = 0$ by differential equivalence. For each V in F, let $K(V) = K_1(V) + K_2(V)$. □

We now prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM 4.1. By Theorem 3.1 it follows that there is an element ρ of M such that if V is in F, then

$$(4.3) \quad \int_V \left| \left(\int_I Q\{\xi_1(J), \xi_2(J)\} \right) - \rho(I) \right| = \inf \left\{ \int_V \left| \left(\int_I Q\{\xi_1(J), \xi_2(J)\} \right) - \eta(I) \right| : \eta \text{ in } M \right\}.$$

We shall freely use various immediate consequences of differential equivalence in this argument.

From Lemma 4.3 it follows that there is a real nonnegative-valued function K with domain F such that $\int_\Omega K(I) = 0$ and if $i = 1,2$, $j = 1,2$, $i \neq j$ and V is in F, then $|\xi_i(V) - \lambda_i(V)| \leq |\xi_i(V) - \lambda_j(V)| + K(V)$. Suppose that V is in F. There is $i = 1,2$ such that $\xi_i(V) = Q\{\xi_1(V), \xi_2(V)\}$. It therefore follows from Lemma 4.2 that

$$
\begin{aligned}
(4.4)\quad & |Q\{\xi_1(V),\xi_2(V)\} - Q\{\lambda_1(V),\lambda_2(V)\}| \leq |\xi_i(V) - \lambda_i(V)| + K(V) \\
&\leq [||\xi_i(V) - \lambda_i(V)| - \int_V |\xi_i(I) - \lambda_i(I)||]_V^3 + \int_V |\xi_i(I) - \lambda_i(I)| + K(V) \\
&\leq [\quad]_V^3 + \int_V |\xi_i(I) - \rho(I)| + K(V) \\
&= [\quad]_V^3 + [(\int_V |\xi_i(I) - \rho(I)| - |\xi_i(V) - \rho(V)|]_V^4 + |\xi_i(V) - \rho(V)| + K(V) \\
&= [\quad]_V^3 + [\quad]_V^4 + |Q\{\xi_1(V),\xi_2(V)\} - \rho(V)| + K(V).
\end{aligned}
$$

Let us note that

$$
(4.5)\qquad \int_\Omega ([\quad]_V^3 + [\quad]_V^4 + K(V)) = 0.
$$

It therefore follows that if W is in in F, then, respectively by (4.3), then (4.4) and (4.5),

$$
\begin{aligned}
&\int_W |Q\{\xi_1(I),\xi_2(I)\} - \rho(I)| \\
&\leq \int_W |Q\{\xi_1(I),\xi_2(I)\} - Q\{\lambda_1(I),\lambda_2(I)\}| \leq \int_W |Q\{\xi_1(I),\xi_2(I)\} - \rho(I)|,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_W \left|\int_I Q\{\xi_1(J),\xi_2(J)\} - \int_I Q\{\lambda_1(J),\lambda_2(J)\}\right| \\
&= \int_W \left|\int_I Q\{\xi_1(J),\xi_2(J)\} - \rho(J)\right| \\
&= \inf\left\{\int_W \left|\left(\int_I Q\{\xi_1(J),\xi_2(J)\}\right) - \eta(I)\right| : \eta \text{ in } M\right\}.
\end{aligned}
$$

5 A Decomposition Theorem

In this section we prove Theorem 5.1, as stated in the introduction. We begin with a lemma.

Lemma 5.1 *Suppose that ξ is in $A(\mathbb{R})(F)$, each of η and ζ is in $A(\xi)$ and β is a function with domain F such that if V is in F, then β_V is η or ζ. Suppose that α is a function with domain F such that if V is in F, then $\alpha(V) = \beta_V(V)$. Then the following statements are true:*

1) *α is in $r(\{\Omega\})(\Sigma)(F)$.*

2) *$\int L(\alpha)$ is in $A(\xi)$, where we let $L = L_{\{\Omega\}}$.*

3) *If $0 < c$, then there is $D << \{\Omega\}$ such that $\sum_D \int_I |(\int_J L(\alpha)) - \beta_I(J)| < c$.*

PROOF. For each I in F, $|\xi(I) - \alpha(I)| \leq |\xi(I) - \eta(I)| + |\xi(I) - \zeta(I)|$.If $E << \{\Omega\}$, then

$$\begin{aligned}|\xi(\Omega) - \sum_E \alpha(I)| &\leq \sum_E |\xi(I) - \alpha(I)| \\ &\leq \sum_E |\xi(I) - \eta(I)| + \sum_E |\xi(I) - \zeta(I)| \\ &\leq \int_\Omega |\xi(I) - \eta(I)| + \int_\Omega |\xi(I) - \zeta(I)|.\end{aligned}$$

Clearly, then, 1) is true.

Again, suppose that $E << \{\Omega\}$. Then

$$\sum_E |\xi(V) - \int_V L(\alpha)(J)|$$

$$\leq \sum_E |\xi(V) - \alpha(V)| + \sum_E |\alpha(V) - L(\alpha)(V)| + \sum_E |L(\alpha)(V) - \int_V L(\alpha)(J)|$$

$$\leq \sum_E |\xi(V) - \alpha(V)| + \sum_E (L(\alpha)(V) - G(\alpha)(V)) + L(\alpha)(\Omega) - \int_\Omega L(\alpha)(J)$$

$$\leq \int_\Omega |\xi(I) - \eta(I)|$$

$$+ \int_\Omega |\xi(I) - \zeta(I)| + L(\alpha)(\Omega) - G(\alpha)(\Omega) + L(\alpha)(\Omega) - \int_\Omega L(\alpha)(J).$$

Therefore 2) is true.

We now show that 3) is true. Suppose that $0 < c$. There is $H << \{\Omega\}$ such that if E is a subset of a refinement of H and μ is η or ζ, then

$$\sum_E \left| \left[\int_I |\mu(J) - \int_J L(\alpha)| \right] - |\mu(I) - \int_I L(\alpha)(J)| \right| < c/4$$

and

$$\sum_E |L(\alpha)(I) - \int_I L(\alpha)(J)| < c/4.$$

Let $N =$ the number of elements in H. For each V in H, there is $E(V) << \{V\}$ such that $L(\alpha)(V) - \sum_{E(V)} \alpha(I) < c/(4N)$, so that

$$\sum_{E(V)} |L(\alpha)(I) - \alpha(I)|$$

$$= \left(\sum_{E(V)} L(\alpha)(I)\right) - \sum_{E(V)} \alpha(I) \leq L(\alpha)(V) - \sum_{E(V)} \alpha(I) < c/(4N).$$

Now, for each V in H, let $E(V)_1 = \{I : I \text{ in } E(V),\ \beta(I) = \eta\}$ and $E(V)_2 = E(V) - E(V)_1$. Then

$$\sum_H \sum_{E(V)} \int_I \left|\left(\int_J L(\alpha)\right) - \beta(I)(J)\right|$$

$$= \sum_H \sum_{E(V)_1} \int_I \left|\left(\int_J L(\alpha)\right) - \eta(J)\right| + \sum_H \sum_{E(V)_2} \int_I \left|\left(\int_J L(\alpha)\right) - \zeta(J)\right|$$

$$\leq \sum_H \sum_{E(V)_1} \left|\left(\int_I L(\alpha)(J)\right) - \eta(I)\right| + \sum_H \sum_{E(V)_2} \left|\left(\int_I L(\alpha)(J)\right) - \zeta(I)\right| + c/2$$

$$\leq \sum_H \sum_{E(V)_1} \left[\left|\left(\int_I L(\alpha)(J)\right) - L(\alpha)(I)\right| + |L(\alpha)(I) - \eta(I)|\right]$$

$$+ \sum_H \sum_{E(V)_2} \left[\left|\left(\int_I L(\alpha)(J)\right) - L(\alpha)(I)\right| + |L(\alpha)(I) - \zeta(I)|\right] + c/2$$

$$< c/4 + \sum_H \sum_{E(V)_1} |L(\alpha)(I) - \eta(I)| + \sum_H \sum_{E(V)_2} |L(\alpha)(I) - \zeta(I)| + c/2$$

$$= 3c/4 + \sum_H \sum_{E(V)} |L(\alpha)(I) - \alpha(I)| < 3c/4 + Nc/(4N) = c.$$

Letting $D = \cup_H E(V)$, we have the desired conclusion.

Lemma 5.2 *Suppose that* $\{p,q,r,s\} \subseteq \mathbb{R}$, $0 \leq K$, $|p-r| \leq |p-s| + K$ *and* $|q-s| \leq |q-r| + K$. *Then* $||p-r| - |q-s|| \leq |p-q| + K$.

Proof. W.l.o.g. assume that $|q-s| \leq |p-r|$. Then

$$||p-r| - |q-s|| = |p-r| - |q-s|$$

$$\leq |p-s| + K - |q-s| \leq |p-s-(q-s)| + K = |p-q| + K.$$

Lemma 5.3 *Suppose that ξ is in $A(\mathbb{R})(F)$, $B << \{\Omega\}$, α is in $r(B)(\Sigma)(F)$ and $\int_\Omega |\xi(I) - \alpha(I)|$ exists. If P is $L_B(\alpha)$ or $G_B(\alpha)$, then $\int P$ is in $A(\xi)$ and for each V in F we get $\int_V |\xi(I) - \int_I P(J)| = \int_V |\xi(I) - \alpha(I)|$.*

PROOF. W.l.o.g., let $L = L_B(\alpha) = P$. There is $D << B$ such that if $E << D$ and a is in $\alpha-$ function on E, then $\sum_E |\xi(J) - a(J)| < 1 + \int_\Omega |\xi(I) - \alpha(I)|$. Suppose that $E << D$. Let $N =$ the number of elements of E. For each I in E there is $H(I) << \{I\}$ and an $\alpha-$ function a on $H(I)$ such that $L(I) - \sum_{H(I)} a(J) < 1/N$, so that $\sum_{H(I)} |L(J) - a(J)| = \sum_{H(I)}(L(J) - a(J)) \leq L(I) - \sum_{H(I)} a(J) < 1/N$, so that

$$\sum_E |\xi(I) - \int_I L| \leq \sum_E \sum_{H(I)} |\xi(J) - \int_J L|$$

$$\leq \sum_E \sum_{H(I)} |\xi(J) - a(J)| + \sum_E \sum_{H(I)} |a(J) - L(J)| + \sum_E \sum_{H(I)} |L(J) - \int_J L|$$

$$< 1 + \int_\Omega |\xi(I) - \alpha(I)| + N/N + L(\Omega) - \int_\Omega L.$$

Therefore $\int L$ is in $A(\xi)$.

Now suppose that V is in F and $0 < c$. There is $D << \{V\}$ such that D is a subset of a refinement of B and such that if $E << D$ and a is an $\alpha-$ function on E, then $|\int_V |\xi - \alpha| - \sum_E |\xi(J) - a(J)|| < c/8$, $|\int_V |\xi - \int L| - \sum_E |\xi(J) - \int_J L|| < c/8$ and $|\int_V L - \sum_E L(J)| < c/8$. Suppose that $E << D$. Let $N =$ the number of elements of E. For each I in E there is $H(I) << \{I\}$ and an $\alpha-$ function a on $H(I)$ such that $L(I) - \sum_{H(I)} a(J) < c/(8N)$,so that $\sum_{H(I)} |L(J) - a(J)| = \sum_{H(I)}(L(J) - a(J)) \leq L(I) - \sum_{H(I)} a(J) < c/(8N)$,so that

$$\left| \int_V |\xi - \int L| - \int_V |\xi - \alpha| \right|$$

$$\leq \left| \int_V |\xi - \int L| - \sum_E \sum_{H(I)} |\xi(J) - \int_J L| \right|$$

$$+ \left| \sum_E \sum_{H(I)} |\xi(J) - \int_J L| - \sum_E \sum_{H(I)} |\xi(J) - L(J)| \right|$$

$$+\left|\sum_E \sum_{H(I)} |\xi(J) - L(J)| - \sum_E \sum_{H(I)} |\xi(J) - a(J)|\right|$$
$$+\left|\sum_E \sum_{H(I)} |\xi(J) - a(J)| - \int_V |\xi - \alpha|\right|$$
$$< c/8 + \sum_E \sum_{H(I)} (L(J) - \int_J L) + \sum_E \sum_{H(I)} (L(J) - a(J)) + c/8$$
$$< c/8 + (\sum_E \sum_{H(I)} L(J)) - \int_V L + Nc/(8N) + c/8 < 4c/8 < c.$$

This establishes the conclusion for $P = L_B(\alpha)$; the argument for $P = G_B(\alpha)$ follows similarly. □

We now prove Theorem 5.1, as stated in the introduction.

PROOF OF THEOREM 5.1 We shall let $Q = \max$. For $i = 1, 2$, there is λ_i' in M such that if V is in F, then $\int_V |\xi_i(I) - \lambda_i'(I)| = \inf\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\}$. Clearly, for $i = 1, 2$, $\xi_i = \int \min\{\xi_i, \int \max\{\xi_1, \xi_2\}\}$. Therefore, for $i = 1, 2$ and each V in F

$$\int_V |\xi_i(I) - \int_I \min\{\lambda_i'(J), \rho(J)\}|$$
$$= \int_V \left|\int_I \min\{\xi_i(J), \int_J \max\{\xi_1, \xi_2\}\} - \int_I \min\{\lambda_i'(J), \rho(J)\}\right|,$$

which, by Theorem 4.1 is

$$\inf\left\{\int_V \left|\int_I \min\{\xi_i(J), \int_J \max\{\xi_1, \xi_2\}\} - \eta(I)\right| : \eta \text{ in } M\right\}$$
$$= \inf\left\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\right\}.$$

Letting, for $i = 1, 2$, $\lambda_i = \int \min\{\lambda_i', \rho\}$, we see that $\rho - \lambda_i$ is in $AB(\mathbb{R})(F)^+$ and for each V in F, $\int_V |\xi_i(I) - \lambda_i(I)| = \inf\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\}$. For $i = 1, 2$, $j = 1, 2$, and $i \neq j$, let δ_i denote the function with domain F given by

$$(\delta_i)_V = \begin{cases} \rho, & \text{if } \xi_i(V) \geq \xi_j(V) \\ \lambda_i, & \text{otherwise.} \end{cases}$$

and γ_i denote the function with domain F given by $\gamma_i(I) = (\delta_i)_I(I)$. By Lemma 5.1, γ_i is in $r(\{\Omega\})(\Sigma)(F)$ and $\int L(\gamma_i)$ is in $A(\xi_i)$.

Now suppose that $i = 1, 2$, $j = 1, 2$, $i \neq j$ and I is in F. If $\xi_i(I) = \xi_j(I)$, then $\max\{\gamma_i(I), \gamma_j(I)\} = \max\{\rho(I), \rho(I)\} = \rho(I)$. If $\xi_i(I) > \xi_j(I)$, then $\max\{\gamma_i(I), \gamma_j(I)\} = \max\{\rho(I), \lambda_j(I)\} = \rho(I)$. Suppose that $i = 1, 2$. We show that for each V in F we have the following existence and equality:

$$\int_V |\xi_i(I) - \gamma_i(I)| = \inf\left\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\right\}.$$

First, by Lemma 4.3, letting $\xi_1^* = \int \max\{\xi_1, \xi_2\}$, $\xi_2^* = \xi_i$, $\lambda_1^* = \rho$ and $\lambda_2^* = \lambda_i$, we see that there is a real-nonnegative-valued function K with domain F such that $\int_\Omega K(I) = 0$ and for each I in F,

$$\left|\left(\int_I \max\{\xi_1(J), \xi_2(J)\}\right) - \rho(I)\right| \leq \left|\left(\int_I \max\{\xi_1(J), \xi_2(J)\}\right) - \lambda_i(I)\right| + K(I),$$

and $|\xi_i(I) - \lambda_i(I)| \leq |\xi_i(I) - \rho(I)| + K(I)$, so that by Lemma 5.2,

$$\left|\left|\left(\int_I \max\{\xi_1(I), \xi_2(I)\}\right) - \rho(I)\right| - |\xi_i(I) - \lambda_i(I)|\right|$$

$$\leq \left|\left(\int_I \max\{\xi_1(I), \xi_2(I)\}\right) - \xi_i(I)\right| + K(I).$$

We shall use the routinely established inequality $||a+b|-|c|| \leq |a|+||b|-|c||$.

Now suppose that $i = 1, 2$, $j = 1, 2$, $i \neq j$ and V is in F. Suppose that $E << \{V\}$. Let $E_1 = \{I : I \text{ in } E,\ \xi_i(I) \geq \xi_j(I)\}$and $E_2 = E - E_1$. Then

$$\left|\sum_E |\xi_i(I) - \gamma_i(I)| - \sum_E |\xi_i(I) - \lambda_i(I)|\right|$$

$$= \left|\sum_{E_1}[|\max\{\xi_1(I), \xi_2(I)\} - \rho(I)| - |\xi_i(I) - \lambda_i(I)|]\right|$$

$$\leq \sum_{E_1} ||\max\{\xi_1(I), \xi_2(I)\} - \int_I \max\{\xi_1(J), \xi_2(J)\}$$

$$+ \left(\int_I \max\{\xi_1(J), \xi_2(J)\} - \rho(I)\right)| - |\xi_i(I) - \lambda_i(I)||$$

$$\leq \Big[\sum_{E_1} |\max\{\xi_1(I), \xi_2(I)\} - \int_I \max\{\xi_1(J), \xi_2(J)\}|\Big]_1$$

$$+\sum_{E_1}\left[\left|\left|(\int_I \max\{\xi_1(J), \xi_2(J)\}) - \rho(I)\right| - |\xi_i(I) - \lambda_i(I)|\right|\right]$$

$$\leq [\quad]_1 + \sum_{E_1}\left[\left|(\int_I \max\{\xi_1(J), \xi_2(J)\}) - \xi_i(I)\right| + K(I)\right]$$

$$= [\quad]_1 + \sum_{E_1}\left|\left(\int_I \max\{\xi_1(J), \xi_2(J)\}\right) - \max\{\xi_1(I), \xi_2(I)\}\right| + \sum_{E_1} K(I)$$

$$\leq 2\sum_{E}\left|\left(\int_I \max\{\xi_1(J), \xi_2(J)\}\right) - \max\{\xi_1(I), \xi_2(I)\}\right| + \sum_{E} K(I).$$

Clearly, these inequalities suffice to show that we have the following existence and equality:

$$\int_V |\xi_i(I) - \gamma_i(I)| = \int_V |\xi_i(I) - \lambda_i(I)| = \inf\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\},$$

so that by Lemma 5.3,

$$\int_V |\xi_i(I) - \int_I L(\gamma_i)(J)| = \int_V |\xi_i(I) - \gamma_i(I)| = \inf\{\int_V |\xi_i(I) - \eta(I)| : \eta \text{ in } M\}.$$

Suppose that $0 < c$. By Lemma 5.1 there is $D << \{\Omega\}$ such that

$$\sum_D \int_I \left|\left(\int_J L(\gamma_i)\right) - (\delta_i)_I(J)\right| < c. \tag{5.1}$$

Let $D_1 = \{I : I \text{ in } D, \delta_i(I) = \rho\}$, $D_2 = D - D_1$ and $W = \cup_{D_1} I$.Then $\rho^{[W]} + \lambda_i^{[\Omega - W]}$ is in M and from (5.1) we have that

$$c > \sum_D \int_I \left|\left(\int_J L(\gamma_i)\right) - (\delta_i)_I(J)\right|$$

$$= \sum_{D_1} \int_I \left|\left(\int_J L(\gamma_i)\right) - \rho(J)\right| + \sum_{D_2} \int_I \left|\left(\int_J L(\gamma_i)\right) - \lambda_i(J)\right|$$

$$= \int_W \left|\left(\int_J L(\gamma_i)\right) - \rho(J)\right| + \int_{\Omega - W} \left|\left(\int_J L(\gamma_i)\right) - \lambda_i(J)\right|$$

$$= \int_\Omega \left|\left(\int_J L(\gamma_i)\right) - [\rho^{[W]} + \lambda_i^{[\Omega - W]}](J)\right|.$$

It therefore follows from Corollary 3.1 that $\int L(\gamma_i)$ is in M.

It remains to be shown that $\rho = \int \max\{\int L(\gamma_1), \int L(\gamma_2)\}$. This is almost immediate, for $\rho = \int \rho = \int \max\{\gamma_1, \gamma_2\} = \int L\max\{\gamma_1, \gamma_2\}$, which by Theorem 2.3 and statement 5) of Theorem 2.1, is $\int \max\{L(\gamma_1), L(\gamma_2)\} = \int \max\{\int L(\gamma_1), \int L(\gamma_2)\}$.

The argument for the case $Q = \min$ is similar to the above, involving mainly a judicious interchange of max and min, a replacement of $\geq$ with $\leq$ in the definition of the δ_is and the replacement of L with G in the final portion of the argument.

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