Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 536–558

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BOREL IMAGES OF SETS OF REALS

Abstract

The main goal of this paper is to generalize several results concerning cardinal invariants to the statements about the associated families of sets. We also discuss the relationship between the additive properties of sets and their Borel images. Finally, we present estimates for the size of the smallest set which is not strongly meager.

1 Introduction

The purpose of this paper is to study additive properties of sets of reals. By reals we mean the Cantor set 2^ω, real line $\mathbb{R}$ or interval $[0,1]$. We will be working in the space 2^ω with addition modulo 2, but most of the results would translate to the interval $[0,1]$ and $\mathbb{R}$.

We will use the standard notation. Let $\mathbb{Q}$ be the canonical countable dense set in the spaces mentioned above. It is the set of rationals in case of $\mathbb{R}$ or $[0,1]$ and a collection of 0-1 sequences that are eventually equal to zero in case of 2^ω.

We will often be using trees on $2^{<\omega}$. A subset $T \subseteq 2^{<\omega}$ is a tree if for every $t \in T$ and $n < |t|$, $t{\restriction}n \in T$. We will also require that T does not have terminal nodes. Let $T{\restriction}n$ denote the n-th level of T and for $n < m$ and $s \in T{\restriction}n$ let

$$\mathsf{succ}_{T,m}(s) = \{t \in T{\restriction}m : s \subseteq t\}.$$

Key Words: null additive, meager additive, strongly meager, strong measure zero
Mathematical Reviews subject classification: Primary: 04A20
Received by the editors May 27, 1993
*Research partially supported by the Israeli Academy of Science, Basic Research Foundation and FONDECYT #1940695, Chile

For a tree T and $s \in T$ let $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$ be the subtree determined by s.

For a set $A \subseteq 2^{<\omega}$ let

$$[A] = \{x \in 2^\omega : \forall n\ x\restriction n \in A \text{ or } x \text{ contains a terminal node from } A\}.$$

Note that in case of a tree, $[T]$ coincides with the set of branches of T. On the other hand, for $s \in 2^{<\omega}$, $[s]$ is the basic open set in 2^ω determined by s.

If $T \subseteq 2^{<\omega}$ is a tree as above and $n \in \omega$ we define the tree $T^{(n)}$ as

$$s \in T^{(n)} \iff |s| < n \text{ or } \exists t \in T\ \big(|s| = |t|\ \&\ s\restriction[n,|s|) = t\restriction[n,|s|)\big).$$

Note that if we identify the set of rationals $\mathbb{Q}$ with elements of 2^ω which are eventually equal to 0, then $[T] + \mathbb{Q} = \bigcup_{n\in\omega}[T^{(n)}]$.

ZFC* always denotes a finite, sufficiently large fragment of ZFC.

Quantifiers $\forall^\infty$ and $\exists^\infty$ denote "for all except finitely many" and "for infinitely many", respectively.

For a set $H \subseteq 2^\omega \times 2^\omega$ and $x, y \in 2^\omega$ let $(H)_x = \{y : \langle x, y\rangle \in H\}$ and $(H)^y = \{x : \langle x, y\rangle \in H\}$. By $\mathcal{M}$ and $\mathcal{N}$ we denote the σ-ideals of meager and Lebesgue measure zero sets respectively.

Definition 1.1 *Suppose that $\mathcal{J}$ is an ideal of subsets of the real line. A Borel set $H \subseteq 2^\omega \times 2^\omega$ is called a $\mathcal{J}$-set if $(H)_x \in \mathcal{J}$ for all $x \in 2^\omega$.*

We say that $X \subseteq 2^\omega$ is an $R^{\mathcal{J}}$ set if for every $\mathcal{J}$-set H,

$$\bigcup_{x\in X}(H)_x \neq 2^\omega.$$

A set $X \subseteq 2^\omega$ is an $SR^{\mathcal{J}}$ set if for every $\mathcal{J}$-set H,

$$\bigcup_{x\in X}(H)_x \in \mathcal{J}.$$

Consider the space 2^ω with group operation $+$ defined as addition modulo 2.

A set X is $\mathcal{J}$-additive if for every set $F \in \mathcal{J}$, $X + F \in \mathcal{J}$ and is not $\mathcal{J}$-covering if for every $F \in \mathcal{J}$, $X + F \neq 2^\omega$.

Let $\mathcal{J}^\star$ denote the ideal of $\mathcal{J}$-additive sets.

Traditionally the sets which are not $\mathcal{N}$-covering are called strongly meager while the sets which are not $\mathcal{M}$-covering are called strong measure zero sets. Let $\mathcal{SZ}$ denote the ideal of strongly measure zero sets and $\mathcal{SM}$ the collection of strongly meager sets.

It is well known that strong measure zero sets form an ideal whereas it is not clear whether the collection strongly meager sets is an ideal.

We have the following easy observation:

Lemma 1.2 *Suppose that $\mathcal{J}$ is a proper, translation invariant, ideal with Borel basis that contains all singletons. Then we have the following inclusions (writing $\rightarrow$ for $\subseteq$):*

$$\begin{tikzcd}
& \textit{not } \mathcal{J}\textit{-covering} & \\
R^{\mathcal{J}} \arrow[ur] & & \mathcal{J}^{\star} \arrow[ul] \\
& SR^{\mathcal{J}} \arrow[ul] \arrow[ur] &
\end{tikzcd} .$$

In particular,

$$\begin{tikzcd}
& S\mathcal{Z} & \\
R^{\mathcal{M}} \arrow[ur] & & \mathcal{M}^{\star} \arrow[ul] \\
& SR^{\mathcal{M}} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S\mathcal{M} & \\
R^{\mathcal{N}} \arrow[ur] & & \mathcal{N}^{\star} \arrow[ul] \\
& SR^{\mathcal{N}} \arrow[ul] \arrow[ur] &
\end{tikzcd} . \square$$

Definition 1.3 *For any proper ideal $\mathcal{J}$ of subsets of X we can define the following cardinal coefficients:*

$\mathsf{add}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \textit{and}\ \bigcup \mathcal{A} \notin \mathcal{J}\}$,
$\mathsf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \textit{and}\ \bigcup \mathcal{A} = X\}$,
$\mathsf{non}(\mathcal{J}) = \min\{|Y| : Y \subseteq X,\ \textit{and}\ Y \notin \mathcal{J}\}$,
$\mathsf{cof}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \textit{and}\ \forall B \in \mathcal{J}\ \exists A \in \mathcal{A}\ B \subseteq A\}$.

Following [20], let $\mathfrak{b}$ and $\mathfrak{d}$ denote the sizes of the smallest unbounded and dominating families in ω^ω respectively.

We have the following lemma:

Lemma 1.4 *If X is an $R^{\mathcal{J}}$ set then every Borel image of X is an $R^{\mathcal{J}}$ set.*
If X is an $SR^{\mathcal{J}}$ set then every Borel image of X is an $SR^{\mathcal{J}}$ set.

PROOF. Suppose that $X \subseteq 2^\omega$ and let $f : 2^\omega \longrightarrow 2^\omega$ be a Borel function.

Let $H \subseteq 2^\omega \times 2^\omega$ be a Borel $\mathcal{J}$-set. Define

$$\widetilde{H} = \{\langle x, y\rangle : \langle f(x), y\rangle \in H\}.$$

It is easy to see that $\widetilde{H}$ is a Borel set and that

$$\bigcup_{x \in X} (H)_{f(x)} = \bigcup_{x \in X} \widetilde{H}_x . \square$$

Note that we have the following easy observation:

Lemma 1.5 *$X \subseteq 2^\omega$ is an $R^{\mathcal{J}}$-set iff for every Borel set $H \subseteq 2^\omega \times 2^\omega$, such that $(H)_x \in \mathcal{J}$ for all $x \in 2^\omega$,*

$$2^\omega \setminus \bigcup_{x \in X} (H)_x \notin \mathcal{J}.$$

PROOF. Implication ($\leftarrow$) is obvious.

On the other hand suppose that G is a Borel set such that $2^\omega \setminus \bigcup_{x \in X} (H)_x \subseteq G \in \mathcal{J}$. Let $\widehat{H} = H \cup (2^\omega \times G)$. Clearly $\widehat{H}$ witnesses that X is not an $R^{\mathcal{J}}$-set. □

Note that

Theorem 1.6 ([9] p. 434) *Suppose that $X \subseteq 2^\omega$.*

If $H \subseteq X \times 2^\omega$ is a Borel set then there exists a Borel set $\widetilde{H} \subseteq 2^\omega \times 2^\omega$ such that $H = \widetilde{H} \cap (X \times 2^\omega)$.

If $f : X \longrightarrow 2^\omega$ is a Borel function then there exists a Borel function $\widetilde{f} : 2^\omega \longrightarrow 2^\omega$ such that $\widetilde{f} = f \restriction X$. □

Finally we will need the following theorem concerning representation of $\mathcal{M}$- and $\mathcal{N}$-sets.

Lemma 1.7 (Fremlin) *Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel set.*

1. *Assume $(H)_x$ is meager for all x. Then there exists a sequence of Borel sets $\{G_n : n \in \omega\} \subseteq 2^\omega \times 2^\omega$ such that*
   (a) *$(G_n)_x$ is a closed nowhere dense set for all $x \in 2^\omega$,*
   (b) *$H \subseteq \bigcup_{n\in\omega} G_n$.*
2. *For every $\varepsilon > 0$ there exists a Borel set $B \subseteq 2^\omega \times 2^\omega$ such that*
   (a) *$H \subseteq B$,*
   (b) *$(B)_x$ is open for every x,*
   (c) *$\mu((B \setminus H)_x) < \varepsilon$ for every x.*

PROOF. For completeness we present a sketch of the proof here.

Let $\mathcal{G}$ be the family of Borel subsets G of $2^\omega \times 2^\omega$ such that $(G)_x$ is open for every $x \in 2^\omega$.

(1) Let $\mathcal{J}$ be the σ-ideal of subsets of the plane generated by Borel sets F such that $(F)_x$ is closed nowhere dense for all x.

Consider the family Σ of subsets E of the plane such that there are two Borel sets G, H of which $G \in \mathcal{G}$, $H \in \mathcal{J}$ and $E \triangle G \subseteq H$. By induction on Borel class we will show that Σ contains all Borel sets. In particular, it follows that all $\mathcal{M}$-sets are in $\mathcal{J}$, which will finish the proof.

Clearly Σ contains all open sets and is closed under countable unions. We want to show that Σ is also closed under complements.

For a set $G \in \mathcal{G}$ let

$$G' = \{\langle x, y\rangle : y \text{ is an interior point of } 2^\omega \setminus (G)_x\}.$$

Note that $(2^\omega \times 2^\omega) \setminus (G \cup G')$ is a set whose vertical sections are closed and nowhere dense. It follows that in order to show that Σ is closed under complements it is enough to check that G' is a Borel set.

Let $\{U_n : n \in \omega\}$ be a recursive enumeration of a countable base for the family of open subsets of 2^ω.

Note that the following are equivalent:

1. $\langle x, y\rangle \in G'$,
2. $\exists n \left(y \in U_n \ \& \ \forall z \ (z \notin U_n \ \vee \ \langle x, z\rangle \notin G)\right) \ (\mathbf{\Pi}^1_1)$,
3. $\exists n \left(y \in U_n \ \& \ \forall m \ (U_n \cap U_m = \emptyset \ \vee \ \exists z \ (z \in U_m \ \langle x, z\rangle \notin G))\right) \ (\mathbf{\Sigma}^1_1)$.

That shows that G' has a $\mathbf{\Delta}^1_1$ definition which means that it is a Borel set.

(2) Let Σ be the collection of all sets H satisfying the conclusion of the theorem. We will show that Σ is a σ-algebra containing all open sets. In particular, all Borel sets are in Σ. It is enough to show that

(i) unions of finitely many rectangles are in Σ,

(ii) if $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ are in Σ then $\bigcup_{n\in\omega} A_n \in \Sigma$,

(iii) if $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ are in Σ then $\bigcap_{n\in\omega} A_n \in \Sigma$.

Condition (i) is clear. To show (ii) fix $\varepsilon > 0$ and let B_n witness that $A_n \in \Sigma$ with $\varepsilon_n = \varepsilon 2^{-n}$. Let $B = \bigcup_{n\in\omega} B_n$.

(iii) Fix $\varepsilon > 0$ and let B_n witness that $A_n \in \Sigma$ with $\varepsilon/2$. For $n \in \omega$ let

$$Z_n = \left\{x : \mu\big((A_n \setminus A)_x\big) < \varepsilon/2\right\}.$$

Put $B = \bigcup_{n\in\omega}\big((Z_n \times 2^\omega) \cap B_n\big)$. $\square$

2 $SR^{\mathcal{M}}$ sets

In this section we will characterize $SR^{\mathcal{M}}$ sets.

We will need the following characterization of meager sets.

Theorem 2.1 ([5] Prop. 9) *For every meager set $F \subseteq 2^\omega$ there exists $x_F \in 2^\omega$ and a strictly increasing function $f_F \in \omega^\omega$ such that*

$$F \subseteq B(f_F, x_F) = \{x \in 2^\omega : \forall^\infty n\ \exists j \in [f_F(n), f_F(n+1))\ x(j) \neq x_F(j)\}.$$

Moreover, $B(f,x) \subseteq B(g,y)$ if and only if

$$\forall^\infty n\ \exists k\ \Big(g(n) \leq f(k) < f(k+1) \leq g(n+1)\ \&$$
$$x{\restriction}[f(k), f(k+1)) = y{\restriction}[f(k), f(k+1))\Big). \square$$

Theorem 2.2 *1. add($\mathcal{M}$) is the least size of any family $F \subseteq \omega^\omega$ such that there are no $r, h \in \omega^\omega$ such that*

$$\forall f \in F\ \forall^\infty n\ \exists k \in [r(n), r(n+1))\ f(k) = h(k),$$

2. X is $\mathcal{M}$-additive iff for every increasing function $f \in \omega^\omega$ there exists $g \in \omega^\omega$ and $y \in 2^\omega$ such that

$$\forall x \in X\ \forall^\infty n\ \exists k\ \Big(g(n) \leq f(k) < f(k+1) \leq g(n+1)\ \&$$
$$x{\restriction}[f(k), f(k+1)) = y{\restriction}[f(k), f(k+1))\Big),$$

3. non($\mathcal{M}^\star$) is the least size of a bounded family $F \subseteq \omega^\omega$ such that there are no $r, h \in \omega^\omega$ such that

$$\forall f \in F\ \forall^\infty n\ \exists k \in [r(n), r(n+1))\ f(k) = h(k).$$

4. add($\mathcal{M}$) $= \min\{$non($\mathcal{M}^\star$)$, \mathfrak{b}\}$.

PROOF. (1) Follows readily from the Miller-Truss result that add($\mathcal{M}$) $=$ $\min\{$cov($\mathcal{M}$)$, \mathfrak{b}\}$ and the fact that cov($\mathcal{M}$) is the least cardinal of any set $F \subseteq \omega^\omega$ such that there is no $h \in \omega^\omega$ such that

$$\forall f \in F\ \exists^\infty n\ h(n) = f(n).$$

See [10] and [3].

(2) Follows from 2.1 and the fact that $B(f,x) + z = B(f, x+z)$.

(3) Similar to (2).

(4) It was proved by Pawlikowski in [13]. It follows from (1) and (3) and the fact that add($\mathcal{M}$) $\leq \mathfrak{b}$. $\square$

We will now characterize $SR^{\mathcal{M}}$ sets.

Theorem 2.3 *A set $X \subseteq 2^\omega$ is an $SR^{\mathcal{M}}$ set iff every Borel image of X is $\mathcal{M}$-additive and every Borel image of X into ω^ω is bounded.*

PROOF. By 1.2, $SR^{\mathcal{M}}$ sets are $\mathcal{M}$-additive. Thus by 1.4, all Borel images of $SR^{\mathcal{M}}$ sets are $\mathcal{M}$-additive.

To show the second part let F be a homeomorphism (Borel isomorphism is enough) between $2^\omega \setminus \mathbb{Q}$ and the set of increasing functions in ω^ω. Let $H = \{\langle x, y\rangle : y \in B(F(x), x)\}$. Suppose that f is a Borel mapping of X into ω^ω such that the family $f(X)$ is unbounded. Without loss of generality we can assume that $f(X)$ consists of increasing functions. Let $Y = F^{-1}(f(X))$. By 1.4, Y is an $SR^{\mathcal{M}}$ set and by 2.1,

$$\bigcup_{x \in Y} (H)_x \notin \mathcal{M}.$$

This contradiction finishes the proof of one implication.

($\leftarrow$) Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel $\mathcal{M}$-set.

Thus, by 1.7(1) we can assume that $H = \bigcup_{n \in \omega} G_n$, where each set G_n has closed nowhere dense sections. Moreover, by combining 2.1 and 1.7(1), we can find a Borel mapping F such that for every $z \in 2^\omega$, $F(z) = \langle f_z, x_z\rangle \in \omega^\omega \times 2^\omega$ is such that $(H)_z \subseteq B(f_z, x_z)$. By the assumption, the family $\{f_z : z \in X\}$ is bounded.

Without loss of generality we can assume that $f_z = f$ for $z \in X$. By 2.2(2), there exists a function $g \in \omega^\omega$ and real $y \in 2^\omega$ such that

$$\forall z \in X\ \forall^\infty n\ \exists k\ \Big(g(n) \le f(k) < f(k+1) \le g(n+1)\ \&$$
$$x_z{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)\Big).$$

It is clear that

$$\bigcup_{z \in X} (H)_z \subseteq B(g, y) \in \mathcal{M},$$

which finishes the proof. □

3 $SR^{\mathcal{N}}$ sets

In this section we characterize $SR^{\mathcal{N}}$ sets.

We start with the following well-known fact:

Theorem 3.1 ([1], [13]) *$add(\mathcal{N})$ is the least size of any family $F \subseteq \omega^\omega$ such that there is no function $S : \omega \longrightarrow [\omega]^{<\omega}$ with $|S(n)| \le n$ for all n, such that*

$$\forall f \in F\ \forall^\infty n\ f(n) \in S(n).$$

non$(\mathcal{N}^\star)$ *is the least size of any bounded family* $F \subseteq \omega^\omega$ *such that there is no function* $S : \omega \longrightarrow [\omega]^{<\omega}$ *with* $|S(n)| \leq n$ *for all* n, *such that*

$$\forall f \in F\ \forall^\infty n\ f(n) \in S(n).$$

In particular, add$(\mathcal{N}) = \min\{$*non*$(\mathcal{N}^\star), \mathfrak{b}\}$. $\square$

Note that the characterization of non$(\mathcal{N}^\star)$ above is an easy corollary of the equivalence between (1) and (3) in the theorem below.

We get a similar characterisation of $\mathcal{N}$-additive sets but the proof is much harder. We will present it here for completeness.

Theorem 3.2 (Shelah [19]) *Let* $X \subseteq 2^\omega$. *The following conditions are equivalent:*

1. X *is* $\mathcal{N}$*-additive,*
2. *for every increasing function* $f \in \omega^\omega$ *there exists a tree* $T \subseteq 2^{<\omega}$ *such that for all* n, $|T{\restriction}n| \leq f(n)$ *and for every* $x \in X$ *there exists* n *such that* $x \in [T^{(n)}]$,
3. *for every increasing function* $f \in \omega^\omega$ *there exists a sequence* $\{I_n : n \in \omega\}$ *such that*
   (a) *for all* n, $I_n \subseteq 2^{[f(n), f(n+1))}$,
   (b) *for all* n, $|I_n| \leq n$,
   (c) $\forall x \in X\ \forall^\infty n\ x{\restriction}[f(n), f(n+1)) \in I_n$.
4. *there exists a function* $g \in \omega^\omega$ *such that for every increasing function* $f \in \omega^\omega$ *there exists a sequence* $\{I_n : n \in \omega\}$ *such that*
   (a) *for all* n, $I_n \subseteq 2^{[f(n), f(n+1))}$,
   (b) *for all* n, $|I_n| \leq g(n)$,
   (c) $\forall x \in X\ \forall^\infty n\ x{\restriction}[f(n), f(n+1)) \in I_n$.

PROOF. (1) $\rightarrow$ (4) The function g that we are looking for will be $g(n) = n2^n$.
Let $f \in \omega^\omega$ be an increasing function such that for all n,

$$f(n+1) \geq 2^{f(0)+\cdots+f(n)+n}.$$

We will start with a construction of a measure zero set having strong combinatorial properties.

Lemma 3.3 *For every* n *there exists a family* $\{A_s : s \in 2^{f(n)}\}$ *such that*

1. *$A_s \subseteq 2^{[f(n),f(n+1))}$ for $s \in 2^{f(n)}$,*
2. *$\mu([A_s]) = 1 - 2^{-n}$ for $s \in 2^{f(n)}$,*
3. *sets $\{x_s + [A_s] : s \in 2^{f(n)}\}$ are probabilistically independent for every family $\{x_s : s \in 2^{f(n)}\} \subseteq 2^\omega$.*

PROOF. Such a family may be constructed in many different ways. Below is one such construction.

Fix a family of sets $\{I_s : s \in 2^{f(n)}\}$ such that

1. $I_s \subseteq [f(n), f(n+1))$ for all s,
2. $I_s \cap I_t = \emptyset$ for $s \neq t$,
3. $|I_s| = n$ for all s.

For $s \in 2^{f(n)}$ let

$$A_s = \left\{t \in 2^{[f(n),f(n+1))} : \exists k \in I_s\ t(k) = 1\right\}. \square$$

Define a sequence of trees $\{T_m : m \in \omega\}$ using the following condition:

$T_m \restriction f(m) = 2^{f(m)}$ and for $n \geq m$ if $s \in T_m \restriction f(n)$ then $\mathrm{succ}_{T_m, f(n+1)}(s) = A_s$.

Note that

$$\mu([T_m]) = \prod_{n=m}^{\infty} (1 - 2^{-n}) \overset{m\to\infty}{\longrightarrow} 1.$$

In particular, the set $H = 2^\omega \setminus \bigcup_{m\in\omega} [T_m]$ has measure zero. Since X is a null additive the set $X + H$ has measure zero. Passing to complements we conclude that the set

$$\bigcap_{x\in X} \left(x + \bigcup_{m\in\omega} [T_m]\right)$$

has positive measure.

Let $T^\star \subseteq 2^{<\omega}$ be a tree such that $\mu([T^\star_s]) > 0$ for all $s \in T^\star$ and

$$[T^\star] \subseteq \bigcap_{x\in X} \left(x + \bigcup_{m\in\omega} [T_m]\right).$$

Note that if $[T^\star] \subseteq \bigcup_{m\in\omega} [T_m + x]$ then there is $s \in T^\star$ and $m \in \omega$ such that $T^\star_s \subseteq T_m + x$.

For $s \in T^\star$ and $m \in \omega$ define

$$X_{s,m} = \{x \in X : T^\star_s \subseteq T_m + x\}.$$

By the above remark $X = \bigcup_{s \in T^\star, m \in \omega} X_{s,m}$.

We will first show that each set $X_{s,m}$ satisfies the conditions of (4).

Fix $\bar{s} \in T^\star, \bar{m} \in \omega$ and let $n \geq \bar{m}, |\bar{s}|$. By extending, if necessary, we can assume that $\mu([T^\star_{\bar{s}}]) > \frac{1}{2}\mu([\bar{s}])$.

We will estimate the size of the set $\{x \restriction f(n) : x \in X_{\bar{s},\bar{m}}\}$. Consider the finite tree

$$\widetilde{T} = \bigcap_{x \in X_{\bar{s},\bar{m}}} \Big((T_{\bar{m}} \restriction f(n+1)) + x \restriction f(n+1) \Big).$$

Observe that $\widetilde{T}$ has height $f(n+1)$ and contains $T^\star_{\bar{s}} \restriction f(n+1)$. Let $\bar{t}$ be any element of $\widetilde{T} \restriction f(n)$. For every element $x \in X_{\bar{s},\bar{m}}$ there is a $t_x \in T_{\bar{m}} \restriction f(n)$ such that $x \restriction f(n) + t_x = \bar{t}$. Since t_x depends only on $x \restriction f(n)$ the sets $\{x \restriction f(n) : x \in X_{\bar{s},\bar{m}}\}$ and $\{t_x : x \in X_{\bar{s},\bar{m}}\}$ have the same size. It follows that

$$\mathsf{succ}_{\widetilde{T},f(n+1)}(\bar{t}) = \bigcap \left\{ \mathsf{succ}_{T_{\bar{m}},f(n+1)}(t_x) + x \restriction f(n+1) : x \in X_{\bar{s},\bar{m}} \right\}.$$

The sequences t_x are all different and therefore sets $\mathsf{succ}_{T_{\bar{m}},f(n+1)}(t_x)$ represent probabilistically independent sets. Thus their translations are also independent thus computing measures (relativised to $[\bar{t}]$) and using the fact that the sets on the right hand side are independent we get

$$\frac{1}{2} < \mu([T^\star_{\bar{t}}]) \leq \mu\Big(\big[\mathsf{succ}_{\widetilde{T},f(n+1)}(\bar{t})\big] \Big) = \left(1 - \frac{1}{2^n}\right)^{|\{x \restriction f(n) : x \in X_{\bar{s},\bar{m}}\}|}.$$

In particular we get that for all $n \geq |\bar{s}|, \bar{m}$,

$$|\{x \restriction f(n) : x \in X_{\bar{s},\bar{m}}\}| \leq 2^n.$$

Let $\{X^k : k \in \omega\}$ be enumeration of the family $\{X_{s,m} : s \in T^\star, m \in \omega\}$. For each $k, m \in \omega$ let $I^k_m = \{x \restriction [f(m), f(m+1)) : x \in X^k\}$. By the above argument

$$\forall k \; \forall^\infty m \; |I^k_m| \leq 2^{m+1}.$$

Let $g(n) = n \cdot 2^{n+1}$ for all n. Define for $n \in \omega$,

$$I_n = \bigcup \left\{ I^k_n : k \leq n, |I^k_n| \leq 2^{n+1} \right\}.$$

It is clear that the sequence $\{I_n : n \in \omega\}$ has the required properties.

Note that we have proved the following:

Lemma 3.4 *Suppose that $f \in \omega^\omega$ is strictly increasing and that for each n we have a disjoint family $\{I_s : s \in 2^{f(n)}\}$ of subsets of $[f(n), f(n+1))$ of size n. Let*

$$H = \{x \in 2^\omega : \exists^\infty n\ \forall j \in I_{x \restriction f(n)}\ x(j) = 0\}.$$

Then H has measure zero and if X is any set such that $X + H$ has measure zero then there are sets $J_n \subseteq 2^{f(n)}$ such that $|J_n| \leq n2^n$ and

$$\forall x \in X\ \forall^\infty n\ x \restriction f(n) \in J_n. \ \square$$

(4) $\rightarrow$ (3) Suppose that the function $f \in \omega^\omega$ is given. Apply (4) to the function $f'(n) = f(g(n))$ for $n \in \omega$.

Implication (3) $\rightarrow$ (2) is very easy.

(2) $\rightarrow$ (1) Let $H \subseteq 2^\omega$ be a measure zero set. The following lemma is well-known:

Lemma 3.5 *There exists a sequence $\langle F_n : n \in \omega \rangle$ such that $F_n \subseteq 2^n$ for $n \in \omega$, $\sum_{n=1}^\infty |F_n| \cdot 2^{-n} < \infty$ and $H \subseteq \{x \in 2^\omega : \exists^\infty n\ x \restriction n \in F_n\}$.* $\square$

By 3.5, there exists a sequence $\langle F_n : n \in \omega \rangle$ such that $F_n \subseteq 2^n$ for $n \in \omega$, $\sum_{n=1}^\infty |F_n| \cdot 2^{-n} < \infty$ and $H \subseteq \bigcap_{m \in \omega} \bigcup_{k > m} [F_k]$.

Let $f \in \omega^\omega$ be an nondecreasing function with $\lim_{n \to \infty} f(n) = \infty$ such that

$$\sum_{n=1}^{\infty} \frac{f(n) \cdot |F_n|}{2^n} < \infty.$$

Let T be a tree from 3.2(2) for this function. Note that $|T^{(n)} \restriction k| \leq 2^n f(k)$ for all $n, k \in \omega$.

We have $X + H \subseteq \bigcup_{n \in \omega} [T^{(n)}] + H$. For every n,

$$[T^{(n)}] + H \subseteq \bigcap_{m \in \omega} \bigcup_{k > m} [T^{(n)}] + [F_k].$$

By the choice of f the measure of $[T^{(n)}] + [F_k]$ is bounded by $2^n \cdot f(k) \cdot |F_k| \cdot 2^{-k}$. Thus

$$\mu\left([T^{(n)}] + H\right) \leq 2^n \cdot \sum_{k=m}^{\infty} \frac{f(k) \cdot |F_k|}{2^k} \xrightarrow{m \to \infty} 0.$$

Since n is arbitrary we conclude that $X + H$ has measure zero which finishes the proof. $\square$

As a consequence we have:

Theorem 3.6 (Shelah [19]) $\mathcal{N}^\star \subseteq \mathcal{M}^\star$.

PROOF. Follows immediately from 3.2 and 2.2. $\square$

Theorem 3.7 *A set $X \subseteq 2^\omega$ is a $SR^{\mathcal{N}}$ set iff every Borel image of X is $\mathcal{N}$-additive and every Borel image of X into ω^ω is bounded.*

PROOF. As in 2.3, we show that the Borel images of $SR^{\mathcal{N}}$ sets are $\mathcal{N}$-additive. To show the second part let F be a homeomorphism between $2^\omega \setminus \mathbb{Q}$ and the set of increasing functions in ω^ω. For an increasing function $f \in \omega^\omega$ let

$$B(f) = \{x \in 2^\omega : \exists^\infty n \ \forall i \leq n \ x(f(n)+i) = 0\}.$$

Clearly, $B(f)$ is a measure zero set. Let $H = \{\langle x, y\rangle : y \in B(F(x))\}$. Suppose that f is a Borel mapping of X into ω^ω such that the family $f(X)$ is unbounded. Without loss of generality we can assume that $f(X)$ consists of increasing functions. Let $Y = F^{-1}(f(X))$. By 1.4, Y is an $SR^{\mathcal{N}}$ set. By a result of Miller ([11], lemma 5),

$$\bigcup_{x \in Y} (H)_x \notin \mathcal{N}.$$

To show the other implication we need the following fact.

Theorem 3.8 (Recław [17]) *Let $X \subseteq 2^\omega$ be a set such that for every Borel function $x \rightsquigarrow f^x \in \omega^\omega$ there exists a function $S : \omega \longrightarrow [\omega]^{<\omega}$ such that $|S(n)| \leq n$ for all n and*

$$\forall x \in X \ \forall^\infty n \ f^x(n) \in S(n).$$

Then X is a $SR^{\mathcal{N}}$ set. $\square$

Now we are ready to finish the proof of the theorem 3.7. Suppose that F is a Borel mapping of X into ω^ω. By the assumption the set $F(X)$ is bounded. Let f be a function such that $F(X)$ is bounded by the function $2^{f(n+1)-f(n)}$. Identify $2^{f(n+1)-f(n)}$ with $2^{[f(n), f(n+1))}$ for all n. In this way we can identify $F(X)$ with a subset of 2^ω. Part (3) of 3.2 and 3.8 conclude the proof. $\square$

Note that the assumptions that Borel images of X into ω^ω are bounded in 2.2 and 3.7 are necessary. It follows immediately from the following theorem:

Theorem 3.9 (Recław [18]) *Assume Martin's Axiom. Then the real line is a Borel image of some $\mathcal{N}$-additive set X.* $\square$

In particular, the set X from 3.9, is $\mathcal{N}$- and $\mathcal{M}$-additive but is neither a $SR^{\mathcal{N}}$- nor $SR^{\mathcal{M}}$ set.

It is also consistent (see [13]) that there exists a set $X \notin SR^{\mathcal{N}}$ such that all Borel images of X are null-additive.

4 $R^{\mathcal{M}}$ sets

In this section we will study $R^{\mathcal{M}}$ sets. All the results of this section can be found in [4]. They were also proved by Pawlikowski and Recław in [16] and [15] and Recław in [17].

Definition 4.1 *A set $X \subseteq 2^\omega$ has Rothberger's property (is a C'' set) if for every sequence of open covers of X, $\{\mathcal{G}_n : n \in \omega\}$ there exists a sequence $\{U_n : n \in \omega\}$ with $U_n \in \mathcal{G}_n$ such that $X \subseteq \bigcup_{n\in\omega} U_n$.*

Note that Rothberger's property is the topological version of strong measure zero. We have the following:

Theorem 4.2 (Fremlin, Miller [12]) *The following are equivalent:*

1. *$X \subseteq 2^\omega$ has Rothberger's property*
2. *X has strong measure zero with respect to every metric which gives X the same topology.* $\square$

Let C'' be the collection of subsets of 2^ω which have Rothberger's property. It is easy to see that C'' is a σ-ideal.

Theorem 4.3 (Bartoszyński, Judah, Pawlikowski, Recław [4], [17], [16]) *The following conditions are equivalent:*

1. *X is an $R^{\mathcal{M}}$ set,*
2. *for every Borel function $x \rightsquigarrow f^x \in \omega^\omega$ there exists a function $g \in \omega^\omega$ such that*
$$\forall x \in X\ \exists^\infty n\ f^x(n) = g(n).$$
3. *for every Borel function $x \rightsquigarrow \langle Y^x, f^x\rangle \in [\omega]^\omega \times \omega^\omega$ there exists a $g \in \omega^\omega$ such that*
$$\forall x \in X\ \exists^\infty n \in Y^x\ \ f^x(n) = g(n).$$
4. *Every Borel image of X has Rothberger's property.*

As a consequence we get:

Theorem 4.4 (Bartoszyński, Judah [4])

$$\mathsf{cf}(\mathsf{cov}(\mathcal{M})) \geq \mathsf{add}(R^{\mathcal{M}}) \geq \mathsf{add}(C'') \geq \mathsf{add}(\mathcal{N}).$$

5 $R^{\mathcal{N}}$ sets

In this section we will study $R^{\mathcal{N}}$ sets. Most of the results of this section were obtained independently (and earlier) by Pawlikowski.

Let us start with the following definition.

Definition 5.1 *A set $G \subseteq 2^\omega$ is called small if there exists a sequence of disjoint intervals $\{I_n : n \in \omega\}$ and a sequence $\{J_n : n \in \omega\}$ such that for all $n \in \omega$,*

1. $J_n \subseteq 2^{I_n}$,
2. $|J_n| \cdot 2^{-|I_n|} \leq 2^{-2n}$,
3. $G \subseteq \{x \in 2^\omega : \exists^\infty n \; x{\restriction}I_n \in J_n\}$.

We denote the set $\{x \in 2^\omega : \exists^\infty n \; x{\restriction}I_n \in J_n\}$ by $(I_n, J_n)_{n=0}^\infty$.

For a function $f \in \omega^\omega$ such that $f(n) > 0$ for all n and a series $\sum_{n=0}^\infty \varepsilon_n < \infty$ let

$$\mathcal{X}_f = \prod_{n \in \omega} f(n)$$

and

$$\Sigma_f = \left\{ S \in \left([\omega]^{<\omega}\right)^\omega : \frac{|S(n)|}{f(n)} < \varepsilon_n \right\}.$$

For the rest of this section we will assume that $\varepsilon_n = 2^{-n}$ for all n.

Note that by setting $f(n) = 2^{|I_n|}$ and $S(n) = J_n$ we can identify the set

$$\{x \in 2^\omega : \exists^\infty n \; x{\restriction}I_n \in J_n\}$$

with the set $\{z \in \mathcal{X}_f : \exists^\infty n \; z(n) \in S(n)\}$.

Definition 5.2 *Let $\mathcal{H}$ be the ideal of all sets $X \subseteq 2^\omega$ such that every Borel image of X into ω^ω is bounded.*

By replacing "Borel" by "continuous" we get a weaker property introduced by Hurewicz as E^{**} ([8])

We will start with the following forcing characterization of $R^{\mathcal{N}}$ sets.

Recall that for a model M, $\mathsf{R}(M)$ denotes the set of random reals over M.

Theorem 5.3 *X is an $R^{\mathcal{N}}$-set iff for every countable elementary submodel model $M \prec H(\lambda)$ there exists a real z such that z is random over $M[x]$ for $x \in X$.*

PROOF. Recall that $H(\lambda)$ is the collection of sets hereditarily of size $< \lambda$. If λ is a regular, uncountable cardinal then $H(\lambda)$ is a model for a large fragment of ZFC. Moreover, in this context, $M[x]$ is a "closure" of $M \cup \{x\}$.

($\leftarrow$) Let H be an $\mathcal{N}$-set. Choose M containing the code of H and let z be as above. Then $z \notin \bigcup_{x \in X} (H)_x$.

($\rightarrow$) Let $M \models \mathsf{ZFC}^\star$ be a countable model. Consider the set

$$\bar{H} = \{\langle x, y\rangle : y \notin \mathsf{R}(M[x])\}.$$

M is countable and can be coded as a real. Note that $y \notin \mathsf{R}(M[x])$ is equivalent to the statement "there exists a code for a null set $F \in M[x]$ such that $y \in F$". Since M is countable this statement is arithmetical (in M). It follows that $\bar{H}$ is a Borel set. Let z be such that $z \notin (\bar{H})_x$ for $x \in X$. Clearly z is the real we are looking for. $\square$

Theorem 5.4 *Suppose that $X \in \mathcal{H}$. The following are equivalent:*

1. *X is an $R^{\mathcal{N}}$ set,*
2. *for every function $f \in \omega^\omega$ and every Borel mapping $x \rightsquigarrow S^x \in \Sigma_f$ there exists a function $g \in \mathcal{X}_f$ such that*

$$\forall x \in X \ \forall^\infty n \ g(n) \notin S^x(n).$$

PROOF. Implication (1) $\rightarrow$ (2) is very easy because $H = \{(x, g) : \exists^\infty n \ g(n) \in S^x(n)\}$ is a Borel $\mathcal{N}$-set in $X \times \mathcal{X}_f$.

(2) $\rightarrow$ (1) The proof is based on techniques from [2].
Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel $\mathcal{N}$-set.

Lemma 5.5 *There exist increasing interleaved sequences $\{n_k, m_k : k \in \omega\}$ and families $\{J^x_k, \widetilde{J}^x_k : k \in \omega\}$ such that for all $k \in \omega$,*

1. *$J^x_k \subseteq 2^{[n_k, n_{k+1})}$, $\widetilde{J}^x_k \subseteq 2^{[m_k, m_{k+1})}$,*
2. *mappings $x \rightsquigarrow \{J^x_k : k \in \omega\}$ and $x \rightsquigarrow \{\widetilde{J}^x_k : k \in \omega\}$ are Borel,*
3. *$|J^x_k| \cdot 2^{n_k - n_{k+1}} \leq 2^{-2k}$, $|\widetilde{J}^x_k| \cdot 2^{m_k - m_{k+1}} \leq 2^{-2k}$ for all $x \in X$,*
4. *$(H)_x \subseteq \left([n_k, n_{k+1}), J^x_k\right)^\infty_{k=0} \cup \left([m_k, m_{k+1}), \widetilde{J}^x_k\right)^\infty_{k=0}$.*

PROOF. By 1.7(2), we can find a sequence of Borel sets $\{B_n : n \in \omega\}$ such that $(H)_x \subseteq (\bigcap_{n\in\omega} B_n)_x$ for $x \in X$ and

$$\forall x \in X\ \forall^\infty n\ \mu((B_n)_x) < \frac{1}{2^n}.$$

Thus we can work with B_n's rather than with H.

The rest of the proof is the repetition of the proof of theorem 2.2 in [2] (using the fact that $X \in \mathcal{H}$). $\square$

As in [2], define for $k \in \omega$ and $x \in X$ sets

$$S_k^x = \left\{ s \in 2^{[n_k, m_k)} : s \text{ has at least } 2^{n_{k+1}-m_k-k} \text{ extensions in } J_k^x \right\}$$

and for $k > 0$

$$\widetilde{S}_k^x = \left\{ s \in 2^{[n_k, m_k)} : s \text{ has at least } 2^{n_k-m_{k-1}-k} \text{ extensions in } \widetilde{J}_{k-1}^x \right\}.$$

The mapping $x \rightsquigarrow \{S_k^x \cup \widetilde{S}_k^x : k \in \omega\}$ is Borel. Moreover, $\{S_k^x \cup \widetilde{S}_k^x : k \in \omega\} \in \Sigma_f$, where $f(k) = 2^{m_k - n_k}$. Therefore, by the assumption about X there exists a real z such that

$$\forall x \in X\ \forall^\infty k\ z{\restriction}[n_k, m_k) \notin S_k^x \cup \widetilde{S}_k^x.$$

Define for $x \in X$ and $k \in \omega$,

$$T_k^x = \left\{ s \in 2^{[m_k, n_{k+1})} : z{\restriction}[n_k, m_k)^\frown s \in J_k^x \text{ or } s^\frown z{\restriction}[n_{k+1}, m_{k+1}) \in \widetilde{J}_{k+1}^x \right\}.$$

As before we easily check that the assumption about X yields that there exists a real y such that

$$\forall x \in X\ \forall^\infty k\ y{\restriction}[m_k, n_{k+1}) \notin T_k^x.$$

Now define $\widetilde{z} \in 2^\omega$ as

$$\widetilde{z}(n) = \begin{cases} z(n) & \text{if } n \in [n_k, m_k) \text{ for some } k \\ y(n) & \text{if } n \in [m_k, n_{k+1}) \text{ for some } k \end{cases}.$$

As in the proof of 2.2 in [2] we check that

$$\widetilde{z} \notin \bigcup_{x\in X} (H)_x. \ \square$$

Theorem 5.6 (Pawlikowski [14]) *Suppose that $X \in \mathcal{H}$. Then $X \in R^{\mathcal{N}}$ iff for every measure zero set $E \in \mathcal{N}$, $E \cap X \in R^{\mathcal{N}}$.*

PROOF. The implication ($\rightarrow$) is obvious.

To show the other implication consider the space $\mathcal{X}_f = \prod_{n\in\omega} f(n)$ equipped with its standard product measure.

We will work in the space $2^\omega \times \mathcal{X}_f$.

By 5.4 it is enough to show that for every function $f \in \omega^\omega$ and every Borel mapping $x \rightsquigarrow S^x \in \Sigma_f$ there exists a function $g \in \mathcal{X}_f$ such that

$$\forall x \in X\ \forall^\infty n\ g(n) \notin S^x(n).$$

Suppose that a function f and a Borel mapping as above are given. Let $H \subseteq 2^\omega \times \mathcal{X}_f$ be the Borel set such that $(H)_x = \{h \in \mathcal{X}_f : \exists^\infty n\ h(n) \in S^x(n)\}$.

Construct a sequence of elements of $\mathcal{X}_f$, $\{g_n : n \in \omega\}$ and a sequence of measure zero sets $\{E_n : n \in \omega\}$ such that

1. $E_n = \{x \in X : g_n \in (H)_x\}$ for all n,
2. $g_{n+1} \in \mathcal{X}_f \setminus \bigcup\{(H)_x : x \in E_0 \cup \cdots \cup E_n\}$.

The existence of these sequences follows from Fubini's theorem and the assumption about X. Note that the set $\bigcup\{(H)_x : x \in E_0 \cup \cdots \cup E_n\}$ does not have full measure.

For $x \in X$ define $h^x \in \omega^\omega$ as follows:

$$h^x(n) = \min\{k : \forall j > k\ g_n(j) \notin S^x(j)\} \text{ for } n \in \omega.$$

Note that $h^x(n)$ is defined for all except finitely many values of n.

Since the mapping $x \rightsquigarrow h^x$ is Borel we can find an increasing function $h \in \omega^\omega$ such that $h^x \leq^\star h$ for all $x \in X$. Let

$$g(k) = g_n(k) \text{ for } h(n) < k \leq h(n+1).$$

It is easy to see that

$$\forall x \in X\ \forall^\infty n\ g(n) \notin S^x(n)$$

which finishes the proof. □

As a corollary we get:

Theorem 5.7 *Suppose that $X \cap E \in SR^{\mathcal{N}}$ for every $E \in \mathcal{N}$. Then $X \in R^{\mathcal{N}}$.*

PROOF. It is enough to show that $X \in \mathcal{H}$. Suppose that $F : 2^\omega \longrightarrow \omega^\omega$ is a Borel mapping. We can find a sequence of compact sets $\{A_n : n \in \omega\}$ such that

1. $F \restriction A_n$ is continuous,

2. $\bigcup_{n\in\omega} A_n$ has full measure.

Let $E = 2^\omega \setminus \bigcup_{n\in\omega} A_n$. Since $E \cap X \in SR^{\mathcal{N}}$, it follows from 3.7 that $F(E\cap X)$ is bounded. On the other hand, $F(\bigcup_{n\in\omega} A_n)$ is also bounded, which finishes the proof. □

A set $X \subseteq 2^\omega$ is a Sierpiński set if $X \cap H$ is countable for every measure zero set $H \subseteq 2^\omega$.

Theorem 5.8 (Pawlikowski [14]) *Every Sierpiński set is strongly meager.*

PROOF. All countable sets are in $SR^{\mathcal{N}}$ and all $R^{\mathcal{N}}$ sets are strongly meager. □

We do not know if the $R^{\mathcal{N}}$ sets form an ideal. In fact we do not know if $R^{\mathcal{N}} \cap \mathcal{H}$ is an ideal. We only have the following result:

Theorem 5.9 *If $R^{\mathcal{N}} \cap \mathcal{H}$ is an ideal then $R^{\mathcal{N}} \cap \mathcal{H}$ is a σ-ideal.*

PROOF. Suppose that $\{X_n : n \in \omega\}$ is an increasing sequence of elements of $R^{\mathcal{N}} \cap \mathcal{H}$. We will show that $X = \bigcup_{n\in\omega} X_n \in R^{\mathcal{N}}$. We will use 5.4. Let $f \in \omega^\omega$ and suppose that $x \rightsquigarrow S^x \in \Sigma_f$ is a Borel mapping. By the assumption, for every n there exists a function $g_n \in \mathcal{X}_f$ such that

$$\forall x \in X_n \ \forall^\infty k \ g_n(k) \notin S^x(k).$$

As before, for $x \in X$ define $h^x \in \omega^\omega$ as follows:

$$h^x(n) = \min\{k : \forall j > k \ g_n(j) \notin S^x(j)\} \text{ for } n \in \omega.$$

Since mapping $x \rightsquigarrow h^x$ is Borel we can find an increasing function $h \in \omega^\omega$ such that $h^x \leq^\star h$ for all $x \in X$. Let

$$g(k) = g_n(k) \text{ for } h(n) < k < h(n+1).$$

It is easy to see that

$$\forall x \in X \ \forall^\infty n \ g(n) \notin S^x(n)$$

which finishes the proof. □

6 Strongly meager sets

In this section we will estimate the smallest size of a set which is not strongly meager.

We start with the following forcing characterization of strongly meager sets.

Theorem 6.1 *The following are equivalent:*

1. X is strongly meager,

2. for every countable model $M \models ZFC^\star$ there exists s such that $X + s \subseteq R(M)$,

3. for every measure zero set $H \subseteq 2^\omega \times 2^\omega$ there exists s such that

$$\bigcup_{x \in X} (H)_{x+s} \neq 2^\omega.$$

PROOF. (1) $\to$ (3) Suppose that $H \subseteq 2^\omega \times 2^\omega$ has measure zero. Let y be such that $(H)^y$ has measure zero. Since X is strongly meager there exists s such that $(X + s) \cap (H)^y = \emptyset$. In particular, $y \notin \bigcup_{x \in X} (H)_{x+s}$.

(3) $\to$ (2) Let $G = 2^\omega \setminus \mathsf{R}(M)$ and let H be such that $(H)_x = G + x$. Find

$$s \in 2^\omega \setminus \bigcup_{x \in X} (H)_x = 2^\omega \setminus (X + G).$$

Clearly s is the real we are looking for.

(2) $\to$ (1) Suppose that G is a null set. Let M be a countable model such that $G \in M$. Note that if $x + s \in \mathsf{R}(M)$ then $x + s \notin G$. $\square$

Definition 6.2 *For $f \in \omega^\omega$ define the following cardinal invariants:*

$$cov(\Sigma_f) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \Sigma_f \ \& \ \forall g \in \mathcal{X}_f \ \exists S \in \mathcal{A} \ \exists^\infty n \ g(n) \in S(n)\}$$

and

$$cof(\Sigma_f) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \Sigma_f \ \& \ \forall g \in \mathcal{X}_f \ \exists S \in \mathcal{A} \ \forall^\infty n \ g(n) \in S(n)\}.$$

Let $non(\mathcal{SM})$ be the least size of a set which is not strongly meager.

We start with the following fact:

Theorem 6.3 (Shelah, Brendle-Just) *There exists a measure zero set $H \subseteq 2^\omega$ such that for every set Y, if $Y + H$ has measure zero then there is z such that $Y + z \subseteq H$.*

PROOF. We shall describe how to modify the set constructed on page 543 to get the required set H.

Lemma 6.4 *There exists an increasing function $f \in \omega^\omega$ and a sequence $\{U_n : n \in \omega\}$ such that for all $n \in \omega$:*

1. $f(n+1) \geq 2^{f(0)+\cdots+f(n)+n}$,

2. $U_n \subseteq 2^{[f(n),f(n+1))}$,

3.

$$\mu([U_n]) \geq 1 - \left(1 - \frac{1}{2^n}\right)^{f(n)},$$

4. *for every* $X \subseteq 2^{[f(n),f(n+1))}$, $|X| \leq n2^{n+1}$, $X + U_n \neq 2^{[f(n),f(n+1))}$.

PROOF. Suppose that $f(n)$ and U_{n-1} have been chosen. Let k be such that

$$\left(1 - \frac{1}{2^k}\right)^{n2^{n+1}} \geq 1 - \left(1 - \frac{1}{2^n}\right)^{f(n)}$$

and

$$\left(1 - \frac{1}{2^{n+1}}\right)\left(1 - \frac{1}{2^k}\right)^{n2^{n+1}-1} \geq \left(1 - \frac{1}{2^n}\right).$$

Define $f(n+1)$ such that $f(n+1) \geq f(n)+kn2^{n+1}+n2^{f(n)}$. Divide $[f(n), f(n+1))$ into $n2^{n+1}$ pairwise disjoint intervals of size k, say $\{J_i : i \leq n2^{n+1}\}$.

Let

$$U_n = \left\{s \in 2^{[f(n),f(n+1))} : \forall i \leq n2^{n+1}\ \exists j \in J_i\ s(j) = 1\right\} \text{ for } n \in \omega.$$

By the choice of k we have

$$\mu([U_n]) \geq 1 - \left(1 - \frac{1}{2^n}\right)^{f(n)}.$$

Suppose that $X \subseteq 2^{[f(n),f(n+1))}$, $|X| \leq n2^{n+1}$. Let $X = \{x_i : i \leq n2^{n+1}\}$. Define $x^\star$ such that $x^\star \restriction J_i = x_i \restriction J_i$ for every i.

It is easy to see that $x^\star \notin U_n + X$. □

As in 3.3, define sets $\{I_s : s \in 2^{f(n)}\}$ such that

1. $I_s \subseteq [f(n), f(n+1))$ for all s,

2. $I_s \cap I_t = \emptyset$ for $s \neq t$,

3. $|I_s| = n - 1$ for all s,

4. for every $s \in 2^{f(n)}$ there exists $i \leq n2^{n+1}$ such that $I_s \subseteq J_i$.

For $s \in 2^{f(n)}$ let
$$A_s = \{t \in U_n : \exists k \in I_s\ t(k) = 1\}.$$

Note that the choice of k and (1)-(4) above guarantee that the sets A_s satisfy the conditions of 3.3 (except that the measure of $[A_s]$ is different).

Let T_m's and H be defined as in 3.2.

Let $\widehat{T} = \prod_{n \in \omega} U_n$. Note that $\bigcup_m [T_m] \subseteq \mathbb{Q} + \widehat{T}$.

As in the proof of 3.2, we show that if Y is a set such that $Y + H$ has measure zero then there exists a sequence $\{I_n : n \in \omega\}$ such that

1. for all n, $I_n \subseteq 2^{[f(n), f(n+1))}$,
2. for all n, $|I_n| \leq n2^{n+1}$,
3. $\forall x \in Y\ \forall^\infty n\ x \restriction [f(n), f(n+1)) \in I_n$.

Note that if Y has the above property then $Y + \widehat{T} + \mathbb{Q} \neq 2^\omega$. This is guaranteed by 6.4(4). Let $z \in 2^\omega \setminus (Y + \widehat{T} + \mathbb{Q})$. Clearly

$$z + Y \subseteq 2^\omega \setminus (\widehat{T} + \mathbb{Q}) \subseteq 2^\omega \setminus \bigcup_m [T_m] = H,$$

which finishes the proof. □

Now we can characterize strongly meager sets.

Theorem 6.5 $\min_f \mathit{cov}(\Sigma_f) \leq \mathit{non}(\mathcal{SM}) \leq \min_f \mathit{cof}(\Sigma_f)$.

PROOF. The first inequality is proved like 5.4 (see [2] for details).

Fix $f \in \omega^\omega$. Let $\mathcal{A} \subseteq \Sigma_f$ be a family of size $\mathrm{cof}(\Sigma_f)$ such that

$$\forall g \in \mathcal{X}_f\ \exists S \in \mathcal{A}\ \forall^\infty n\ g(n) \in S(n).$$

By increasing f and doing some elementary coding we can assume that there is a function $\tilde{f}$ such that $f(n) = 2^{\tilde{f}(n+1) - \tilde{f}(n)}$ for all n. For $S \in \Sigma_f$ define

$$P_S = \left\{x \in 2^\omega : \forall^\infty n\ x \restriction [\tilde{f}(n), \tilde{f}(n+1)) \in S(n)\right\},$$

where $S(n)$ is treated as a subset of $2^{[\tilde{f}(n), \tilde{f}(n+1))}$. Let H be the set from 6.3.

Note that 6.4 gives us enough freedom to ensure that $P_S + H$ has measure zero for all $S \in \Sigma_f$. This is because in 6.4(4) we can replace the clause $|X| < n2^{n+1}$ by $|X| < g(n)$, where g is an arbitrary fixed function. (cf. [6], lemma 9).

For each $S \in \mathcal{A}$ choose $x_S \in 2^\omega$ such that $P_S + x_S \subseteq H$. We claim that $X = \{x_S : S \in \mathcal{A}\}$ is not strongly meager and show that $X + H = 2^\omega$.

Suppose that $y \in 2^\omega \setminus (X + H)$. Let $S \in \mathcal{A}$ be such that $y \in P_S$. Now $y \in P_S \subseteq H + x_S \subseteq H + X$. Contradiction. $\square$

A forcing notion $\mathcal{P}$ has Laver property if it does not "increase" $\mathsf{cof}(\Sigma_f)$. In other words, if $\mathcal{P}$ has Laver property then for every $g \in \mathbf{V}^{\mathcal{P}} \cap \mathcal{X}_f$ there exists $S \in \mathbf{V} \cap \Sigma_f$ such that $g(n) \in S(n)$ for all n.

In this language we can formulate the previous theorem as follows:

Theorem 6.6 *Suppose that $\mathbf{V}'$ is a generic extension of $\mathbf{V}$ obtained by a forcing notion which has Laver property. Then the set $\mathbf{V} \cap 2^\omega$ is not strongly meager in $\mathbf{V}'$.* $\square$

Acknowledgments

We would like to thank Joerg Brendle for reading the manuscript and for his helpful remarks. We would also like to thank the referees of this paper for shaping the final version of this paper.

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