RESEARCH ARTICLES

Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 527–535

Solomon Leader, Mathematics Department, Hill Center, Busch Campus, Rutgers University, New Brunswick, NJ 08903

CONVERSION FORMULAS FOR THE LEBESGUE-STIELTJES INTEGRAL

0. Introduction

We present here some formulas converting Lebesgue-Stieltjes integrals with a continuous integrator h of bounded variation into Lebesgue integrals over the range of h. A special case is Banach's indicatrix formula displayed as (10) below. Indeed, we use an extension of Banach's formula to prove the general conversion formulas. One of these formulas (6) in conjunction with (8) and the Fubini theorem for the generalized Riemann integral [11] provides a handy proof of Green's theorem. In general all our integrals are defined by Kurzweil-Henstock integration using endpoint tags in the approximating sums. [3], [4], [5] ,[6], [7], [10] and [11] But wherever we have absolute integrability our integrals here are equivalent to Lebesgue-Stieltjes integrals. Our use of differentials is based on the concepts introduced in [6], [7] and [8].

We begin with some relevant definitions. A *cell* is a closed interval $K = [a, b]$ in $\mathbb{R}$ with $a < b$. A *figure* is a finite union of disjoint cells. The *indicator* 1_E of a subset E of $\mathbb{R}$ is the function on $\mathbb{R}$ with value 1 on E and 0 on the complement $\mathbb{R} \setminus E$. E^0 is the interior of E. For h a function on $K = [a, b]$ we define $\Delta h(K) = h(b) - h(a)$. For h continuous and of bounded variation on K, a subset E of K is *dh-measurable* if the differential $1_E\, dh$ is integrable over K. This is equivalent to the existence of the Lebesgue-Stieltjes integral $\int_E dh$. A dh-measurable set differs from a Borel set by a dh-null set. (See [7].) The *variation function* for h is the function v defined on K by

$$v(t) = \int_a^t |dh(s)|. \tag{0}$$

v is characterized by the conditions $v(a) = 0$ and $dv = |dh|$.

Mathematical Reviews subject classification: 26A39
Received by the editors March 5, 1992

1 The Conversion Formulas

Theorem 1 *Let h be a continuous function of bounded variation on $K = [a, b]$ with variation function v. Let f be a function on K with $f\,dh$ absolutely integrable. (That is, the Lebesgue-Stieltjes integral $\int_K f\,dh$ exists and is finite.) For each subset E of K the function F_E given by*

$$F_E(y) = \sum_{t \in E \cap h^{-1}(y)} f(t) \tag{1}$$

is defined and finite for almost all y in $\mathbb{R}$ since the set $h^{-1}(y)$ is finite for almost all y. If E is dh-measurable then F_E is Lebesgue-integrable and

$$\int_{-\infty}^{\infty} F_E(y)dy = \int_E f\,dv. \tag{2}$$

Let A, B be subsets of K satisfying the Hahn conditions

$$A \cap B = \emptyset \quad \text{and} \quad dh = (1_A - 1_B)dv. \tag{3}$$

Then A, B are dh-measurable. Moreover, in terms of (1)

$$\int_{-\infty}^{\infty} F_A(y)dy = \int_K f(dh)^+, \tag{4}$$

$$\int_{-\infty}^{\infty} F_B(y)dy = \int_K f(dh)^-, \tag{5}$$

and for $F = F_A - F_B$

$$\int_{-\infty}^{\infty} F(y)dy = \int_K f\,dh \tag{6}$$

and

$$\int_D F(y)dy = \int_{h^{-1}(D)} f\,dh \text{ for every Borel set } D. \tag{7}$$

The conditions (3) hold if and only if modulo dv-null sets, A is the set of all t in K where $\frac{dh}{dv}(t) = 1$, B the set where $\frac{dh}{dv}(t) = -1$. So for almost all y in $\mathbb{R}$

$$F(y) = \sum_{t \in h^{-1}(y)} \frac{dh}{dv}(t) f(t). \tag{8}$$

PROOF. We ignore the trivial case h constant, $v = 0$. So $h(K)$ is a cell. For $u = \frac{1}{2}(v+h)$ and $w = \frac{1}{2}(v-h)$ we have integrable differentials $du = (dh)^+$ and $dw = (dh)^-$ since $dv = |dh|$. (See [6], [7] and [8].) The Hahn decomposition theorem ([7], Theorem 25) yields A and B satisfying (3). For such A, B we have $1_A\, dh = du$ and $1_B\, dh = -dw$. So A and B are dh-measurable. Since $dh = (1_A - 1_B)dv$ the differentiation theorem ([7], Theorems 17) gives the narrow-sense derivative $\frac{dh}{dv}(t) = 1_A(t) - 1_B(t)$ for dv-all t in K. So

$$\frac{dh}{dv}(t) = \pm 1 \tag{9}$$

for dv-all t, and $dh = \frac{dh}{dv}\, dv$. Let S be the set of all t in K where (9) fails to hold. Its image $h(S)$ is Lebesgue-null since S is dv-null and h is nonexpansive relative to v. Indeed, for *any* function g on K, if S is dg-null, then $g(S)$ is Lebesgue-null by [9], Theorem 2. Therefore for almost all y in $\mathbb{R}$ (9) holds at every t in $h^{-1}(y)$. For such y the compact set $h^{-1}(y)$ must be finite since no t satisfying (9) can be a limit point of $h^{-1}(y)$. Thus, (1) yields (8) for $F = F_A - F_B$. Moreover, the *Banach indicatrix* $N_K(y)$, the number of points t in K such that $h(t) = y$, is finite for almost all y. Indeed, Banach proved ([1], Theorem 2) that N_K is integrable and its integral

$$\int_{-\infty}^{\infty} N_K(y)dy = \int_K dv. \qquad \text{(See also [12] and [13].)} \tag{10}$$

Note that N_K vanishes on the exterior of the cell $h(K)$. So its integral over $\mathbb{R}$ in (10) is just its integral over $h(K)$. Theorem 10 in [9] extends (10) to give

$$\int_D N_K(y)dy = \int_{h^{-1}(D)} dv \text{ for every Borel set } D. \tag{11}$$

We shall prove (2) first for $E = K$, namely

$$\int_{-\infty}^{\infty} F_K(y)dy = \int_K f\, dv. \tag{12}$$

Define the function G on $\mathbb{R}$ by

$$G(y) = \int_{h^{-1}(-\infty,y]} f\, dv. \tag{13}$$

G is constant on each of the two open half-lines whose union is the exterior of the cell $h(K) = [y_0, y_1]$. Specifically, G equals 0 on $(-\infty, y_0)$, $\int_K f\, dv$ on (y_1, ∞). We contend that G is absolutely continuous, and that its derivative $G' = F_K$ almost everywhere. This suffices to give (12) since $G' = F_K = 0$

outside $h(K)$, so $\int_{-\infty}^{\infty} F_K(y)dy = \int_{h(K)} G'(y)dy = G(y_1) - G(y_0) = \int_K f\,dv - 0 = \int_K f\,dv$.

To prove absolute continuity of G let $\varepsilon > 0$ be given. We seek $\delta > 0$ such that

(14) $$\Big|\int_D dG\Big| < \varepsilon \text{ for every figure } D \text{ in } \mathbb{R} \text{ of Lebesgue measure } m(D) < \delta.$$

By (11) for $D = y$ we get $h^{-1}(y)$ dv-null for every y. So we can replace the closed half-line $(-\infty, y]$ in (13) by the open half-line $(-\infty, y)$. Consequently

(15) $$\int_D dG = \int_{h^{-1}(D)} f\,dv \text{ for every figure } D \text{ in } \mathbb{R}$$

since it holds for every cell, and D is a finite union of disjoint cells. Take $\alpha > 0$ small enough so that the absolute continuity of the integral gives

(16) $$\int_C |f|dv < \varepsilon \text{ for every Borel set } C \text{ such that } \int_C dv < \alpha.$$

Similarly use (11) to get $\delta > 0$ small enough so that

(17) $$\int_{h^{-1}(D)} dv < \alpha \text{ for every Borel set } D \text{ such that } m(D) < \delta.$$

Apply (17), (16) with $C = h^{-1}(D)$, and (15) to get (14). So G is absolutely continuous. Thus, $G'(y)$ exists and is finite almost everywhere, and $dG(y) = G'(y)dy$.

Let T be the set of all y in $\mathbb{R}$ such that $G'(y)$ exists, and $h^{-1}(y)$ is a finite set in which every member t satisfies the four conditions: $a < t < b$, $\frac{dh}{dy}(t) = \pm 1$, $\frac{d}{dv}\int_a^t f(s)dv(s) = f(t)$, and $\frac{d}{dv}\int_a^t |f(s)|dv(s) = |f(t)|$. So almost all y in $\mathbb{R}$ belong to T since the h-image of a dv-null set is Lebesgue-null. We contend that $G'(y) = F_K(y)$ for all y in T. That is, in terms of (1),

(18) $$G'(y) = \sum_{t \in h^{-1}(y)} f(t) \quad \text{for all } y \text{ in } T.$$

This is trivial for $h^{-1}(y)$ empty since both sides of (18) vanish if y is not in $h(K)$. Given y in T with $h^{-1}(y)$ nonempty let $c_1, \ldots, c_n$ be the members of $h^{-1}(y)$. Consider any $\varepsilon > 0$ small enough so that the cells $K_i = [c_i - \varepsilon, c_i + \varepsilon]$ for $i = 1, \ldots, n$ are disjoint and lie in K. For all $\delta > 0$ let $D_\delta = [y, y + \delta]$. The nest of compact sets $h^{-1}(D_\delta)$ indexed by $\delta > 0$ has intersection $h^{-1}(y)$.

Therefore, since $h^{-1}(y)$ lies in the neighborhood $K_1^0 \cup \cdots \cup K_n^0$, so does $h^{-1}(D_\delta)$ for δ sufficiently small. Choose such a δ and let $D = D_\delta$. So

$$h^{-1}(D) \subseteq K_1^0 \cup \cdots \cup K_n^0 \text{ with } D = [y, y+\delta]. \tag{19}$$

Consider any $i = 1, \ldots, n$. By (19) h maps the endpoints of K_i into the complement $(-\infty, y) \cup (y+\delta, \infty)$ of D. By the intermediate value theorem the two endpoints of K_i are mapped into opposing half-lines since (9) holds at $t = c_i$ and c_i is the only member of $h^{-1}(y)$ in K_i. So the cell $h(K_i)$ contains D. In particular it contains $y+\delta$. Thus, the compact set $K_i \cap h^{-1}(y+\delta)$ of points t in K_i with $h(t) = y+\delta$ is nonempty. It therefore has a first point s_i and a last point t_i. Clearly $c_i < s_i \leq t_i < c_i + \varepsilon$ if $\frac{dh}{dv}(c_i) = 1$ and $c_i - \varepsilon < s_i \leq t_i < c_i$ if $\frac{dh}{dv}(c_i) = -1$. In the former case take $I_i = [c_i, s_i]$ and $J_i = [c_i, t_i]$. In the latter case take $I_i = [t_i, c_i]$ and $J_i = [s_i, c_i]$. In either case $I_i \subseteq J_i \subset K_i$ and the cells I_i, J_i have a common endpoint c_i at the center of K_i. Since $h(s_i) = h(t_i) = y+\delta$ and $h(c_i) = y$, $\Delta h(I_i) = \Delta h(J_i) = \pm\delta$. Since the endpoint c_i of I_i is the only member of $h^{-1}(y)$ in K_i, it is the only member of $h^{-1}(y)$ in the subset I_i of K_i. The other endpoint of I_i is the only member of $h^{-1}(y+\delta)$ in I_i since I_i abuts the convex closure $[s_i, t_i]$ of $K_i \cap h^{-1}(y+\delta)$. So $h(I_i) = D$ by the intermediate value theorem for the continuous function h. We therefore have $I_i \subseteq E_i$ for $E_i = K_i \cap h^{-1}(D)$. Now $K_i - J_i$ consists of two components. The component which abuts J_i at c_i is mapped by h into $(-\infty, y)$. The component which abuts J_i at its other end is mapped by h into $(y+\delta, \infty)$. So E_i, the set of all t in K_i with $h(t)$ in D, is contained in J_i. By (15), $\Delta G(D) = \int_D dG = \int_{h^{-1}(D)} f\,dv = \sum_{i=1}^n \int_{E_i} f\,dv$ since $h^{-1}(D) = \sum_{i=1}^n E_i$ by (19). Hence, $|\Delta G(D) - \sum_{i=1}^n \int_{I_i} f\,dv| = |\sum_{i=1}^n \int_{E_i - I_i} f\,dv| \leq \sum_{i=1}^n \int_{J_i - I_i} |f|\,dv$ since $I_i \subseteq E_i \subseteq J_i$. Divide through by δ to get

$$\left|\frac{G(y+\delta) - G(v)}{\delta} - \sum_{i=1}^n \frac{1}{\delta}\int_{I_i} f\,dv\right| \leq \sum_{i=1}^n \frac{1}{\delta}\left[\int_{J_i} |f|\,dv - \int_{I_i} |f|\,dv\right]. \tag{20}$$

As ε goes to 0 both s_i and t_i converge to c_i. So I_i and J_i approach their common endpoint c_i. Thus, both $\frac{1}{\delta}\Delta v(I_i)$ and $\frac{1}{\delta}\Delta v(J_i)$ converge to 1 since $\Delta v \geq 0$, $\delta = |\Delta h(I_i)| = |\Delta h(J_i)|$, and $|\frac{dh}{dv}(c_i)| = 1$. So $\frac{1}{\delta}\int_{I_i} f\,dv$ converges to $f(c_i)$ and both $\dfrac{1}{\delta}\int_{J_i} |f|\,dv$ and $\dfrac{1}{\delta}\int_{I_i} |f|\,dv$ converge to $|f(c_i)|$. The right side of (20) therefore converges to 0. Since δ goes to 0 by continuity of h, the left side of (20) converges to $|G'(y) - \sum_{i=1}^n f(c_i)|$. So $G'(y) = \sum_{i=1}^n f(c_i)$ giving (18). This completes the proof of (12).

Given a dh-measurable set E apply (12) with f replaced by $1_E f$. F_K in (12) is thereby replaced by F_E giving (2). Given A, B satisfying (3) apply (2) with $E = A$ to get (4), $E = B$ to get (5), noting that $1_A dv = 1_A dh = (dh)^+$ and $1_B dv = -1_B dh = (dh)^-$. Subtract (5) from (4) to get (6). To get (7) apply (6) with f replaced by $1_{h^{-1}(D)} f$ thereby replacing F by $1_D F$. □

A, B in (3) are unique modulo dv-null sets. The most primitive way to get (3) is to let A consist of all "dh-positive", B all "dh-negative", points in K where t is *dh-positive (dh-negative)* if $\Delta h(I) > 0$ (respectively, $\Delta h(I) < 0$) for all sufficiently small cells I in K with t as an endpoint.

2 The Indicatrix Formulas

As a corollary to Theorem 1 we get the following generalization of Banach's indicatrix theorem. ([1], Theorem 2)

Theorem 2 *Let h be a continuous function of bounded variation on $K = [a, b]$ with variation function v. For E a subset of K let $N_E(y)$ be the number of points t in E such that $h(t) = y$. Then $N_E(y) < \infty$ for almost all y. Moreover, if E is dh-measurable, then N_E is Lebesgue-integrable and*

$$\int_{-\infty}^{\infty} N_E(y)dy = \int_E dv. \tag{21}$$

If A, B satisfy the Hahn conditions (3), then N_A and N_B are Lebesgue-integrable and

$$\int_{-\infty}^{\infty} N_A(y)dy = \int_K (dh)^+, \tag{22}$$

$$\int_{-\infty}^{\infty} N_B(y)dy = \int_K (dh)^-. \tag{23}$$

Finally

$$\int_{h(a)}^{h(b)} 1_D(y)dy = \int_{h^{-1}(D)} dh \text{ for every Borel set } D. \tag{24}$$

PROOF. Apply Theorem 1 with $f = 1$. Then $F_E = N_E$ by (1). So (21), (22), (23) follow respectively from (2), (4), (5). We contend that (24) follows similarly from (7). Recall that for almost all y the set $h^{-1}(y)$ is finite and all its members satisfy (9). By the intermediate value theorem the sign of $\frac{dh}{dv}(t)$ alternates as t advances through the points of $h^{-1}(y)$. So for almost all y (8) with $f = 1$ reduces to $F(y) = \operatorname{sgn}[h(b) - h(a)]$ if y is interior to the interval L with endpoints $h(a)$ and $h(b)$, and to $F(y) = 0$ if y is exterior to L. So

$\int_D F(y)dy = \int_L \operatorname{sgn}[h(b)-h(a)]1_D(y)dy = \int_{h(a)}^{h(b)} 1_D(y)dy$. This with (7) gives (24). □

For $E = K$ (21) is just Banach's formula (10). (The sum of (22) and (23) also gives (10) since $N_A + N_B = N_K$ almost everywhere.) More generally, (21) with $E = h^1(D)$ gives (11) since $N_{h^{-1}(D)} = 1_D N_K$. The special case $D = \mathbb{R}$ which reduces (7) to (6) in Theorem 1 reduces (24) to the fundamental identity $\Delta h(K) = \int_K dh$ which holds for every function h on K. (See [6] and [7].) Note that (24) for the case $h(a) = h(b)$ gives $\int_{h^{-1}(D)} dh = 0$ for every Borel set D in $\mathbb{R}$.

3 An Application to the Proof of Green's Theorem

To show the utility of the conversion formulas we shall apply (6) and (8) of Theorem 1 to get a simple proof of Green's theorem. The conditions in our hypothesis are in several respects stronger than necessary for the validity of our proof. Moreover, the proof is more straightforward than the usual proofs. (e.g. [2])

Theorem 3 *Let D be a closed topological disk in the (x,y)-plane with the boundary of D a rectifiable Jordan curve C. Let p and q be continuous functions on D which with the exception of countably many points have finite partial derivatives q_x and p_y on the interior D^0 of D. Moreover, let q_x and p_y be generalized Riemann-integrable on D. Then*

$$\int\int_D (q_x - p_y)dx\,dy = \oint_C p\,dx + q\,dy \tag{25}$$

for C positively oriented.

PROOF. Parameterize the positively oriented C by $(g(t), h(t))$ where g and h are continuous functions of bounded variation on $K = [a, b]$ with $g(a) = g(b)$ and $h(a) = h(b)$. Since C is rectifiable it is Lebesgue-null in the plane. Thus, since q_x exists and is finite at all but countably many points of D^0, q_x exists almost everywhere in D. The Fubini theorem [11] and/or [4] gives

$$\int\int_D q_x(x,y)dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_D(x,y)q_x(x,y)dx\,dy \tag{26}$$

with the given existence of the generalized Riemann integral on the left implying the existence of the iterated integral on the right. The inner integral on the right exists for almost all y. We contend that for almost all y this integral is just $F(y)$ in (8) of Theorem 1 for the function f given by

$$f(t) = q(g(t), h(t)). \tag{27}$$

$\int_K dh = 0$. So, according to Theorem 1, for almost all y in $h(K)$ the parametrized Jordan curve C crosses the horizontal line $Y = y$ an even number $2n$ of times. Moreover, C intersects this line in a nonempty, finite set of points $(x_1, y), (x_2, y), \ldots, (x_j, y), \ldots, (x_{2n}, y)$ where $x_1 < \cdots < x_j < \cdots < x_{2n}$ and for t_j such that $g(t_j) = x_j$ and $h(t_j) = y$ we have $\frac{dh}{dv}(t_j) = (-1)^j$ for $j = 1, \ldots, 2n$ with v the variation function (0) for h. The intersection of the line $Y = y$ with D has components $S_1, \ldots, S_i, \ldots, S_n$ where the segment S_i has left endpoint (x_{2i-1}, y) and right endpoint (x_{2i}, y). Since q is continuous on S_i and q_x exists and is finite at all but countably many points (x, y) in S_i, the fundamental theorem of calculus ([7], Theorem 17) gives

(28) $$\int_{x_{2i-1}}^{x_{2i}} q_x(x, y)dx = q(x_{2i}, y) - q(x_{2i-1}, y) \text{ for } i = 1, \ldots, n.$$

(See also [11].) Now (8) under (27) gives $F(y) = \sum_{j=1}^{2n}(-1)^j f(t_j) = \sum_{j=1}^{2n}(-1)^j q(x_j, y)$ which is just the sum of (28). So for almost all y

(29) $$\begin{aligned}\int_{-\infty}^{\infty} 1_D(x, y)q_x(x, y)dx &= \sum_{i=1}^{n}\int_{-\infty}^{\infty} 1_{S_i}(x, y)q_x(x, y)dy \\ &= \sum_{i=1}^{n}\int_{x_{2i-1}}^{x_{2i}} q_x(x, y)dx = F(y).\end{aligned}$$

Apply (26), (29), (6) and (27) to get

(30) $$\int\int_D q_x dx\, dy = \int_a^b f(t)dh(t) = \oint_C q\, dy.$$

Apply (30) with p, q interchanged and x, y interchanged. The latter interchange reverses orientation to yield

(31) $$\int\int_D p_y dy\, dx = -\oint_C p\, dx.$$

Subtract (31) from (30) to get (25). □

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