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RESTRICTION THEOREMS IN REAL ANALYSIS

Abstract

This survey article gives a history of the development of the theory concerning restriction theorems in real analysis. The discussion will span the history from Lusin's Theorem through very recent results concerning intersections of Lipschitz, smooth, and Hölder class functions.

1 Introduction

This article is based on an hour lecture given at the Southeastern Section Meeting of the Mathematical Association of America, held at Carson Newman College in Jefferson City, Tennessee, April 8-9, 1994. An expanded version was presented in two talks given at the Joint US-Polish Workshop in Real Analysis, held at the University of Lodz, Poland, July 11-19, 1994. The author acknowledges support of NSF Grant INT-9401673 which allowed him to attend the latter meeting.

Section 2 gives an overview of the types of theorems to be discussed and the notation to be used. Section 3 includes a discussion of the classical restriction theorems, including (1) Lusin's 1912 and 1916 Theorems [30][32] about continuous and derivative restrictions of Lebesgue measurable functions to sets of large measure, (2) Blumberg's 1922 Theorem [3] about continuous restrictions of arbitrary functions to dense sets, and (3) Nikodym's 1929 Theorem [37] about continuous restrictions of Baire measurable functions to residual sets. In Section 4 attention turns to consideration of differentiable and smooth restrictions of continuous functions. The theorems discussed in this section suggest certain differentiable restriction variants of Blumberg's Theorem. Variations on Blumberg's Theorem are discussed in Section 5. Section 6 concerns continuous-, derivative-, and differentiable-restrictions of functions which are Borel-, Lebesgue-, universally-, Baire-, or Marczewski-measurable. Section 7

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is devoted to a discussion of nice restrictions of Lipschitz, smooth, and Hölder class functions. The focus is the recent paper by Olevskiĭ (1994) [39], who gave the final solution to the "Ulam-Zahorski Problem". Section 7 also contains discussion of topological variants of the 1944 $Lip^1 - C^1$ Controlled Intersection Theorem of Federer [18] and the 1951 $C^{(2)} - C^2$ Controlled Intersection of Whitney [47].

Contributions on the part of the current author to various parts of the theory are included, of course. A list of unsolved problems is presented as the discussion unfolds.

2 Overview and Notation

The four types of "Restriction Theorems" that will be considered throughout the article will be described first. Some classes of functions $f : [0,1] \to \mathbb{R}$

$$\dots \to X \to Y \to \dots ,$$

will be given, where the "$\to$" means "$\subseteq$" or "$\Rightarrow$", depending on whether X and Y represent *collections* of functions or the notations for *properties* of functions $f : [0,1] \to \mathbb{R}$.

The discussion will be concerned primarily with the following four types of theorems.

Restriction Theorem. *For every $f \in Y$, there exists a fairly large $N \subseteq [0,1]$ such that the restriction $f|N \in X(rel\ N)$.*

Of course the property which defines the class X must be such that there is some natural interpretation of the property when used to define the class $X(rel\ N)$ of functions with domain N. Better than a simple Restriction Theorem would be the following.

Intersection Theorem. *For every $f \in Y$, there exists a $g \in X$ and a fairly large $N \subseteq [0,1]$ such that $f|N = g|N$.*

This is referred to as an Intersection Theorem because the conclusion is equivalent to saying the the projection of the intersection, $\pi_1(f \cap g)$, contains the set N. Also better than the simple Restriction Theorem would be the following type of theorem which indicates an ability to control the location of the set N to which the function is to be restricted.

Controlled Restriction Theorem. *For every $f \in Y$ and every reasonably large $M \subseteq [0,1]$, there exists a fairly large $N \subseteq M$ such that $f|N \in X(rel\ N)$.*

The strongest of the types of theorems to be considered would be the following.

Controlled Intersection Theorem. *For every $f \in Y$ and every reasonably large $M \subseteq [0,1]$, there exists a $g \in X$ and a fairly large $N \subseteq M$ such that $f|N = g|N$.*

The meanings of the "large" notions will vary. Sometimes it will mean large in cardinality, sometimes it will mean large in some measure theoretic sense, and sometimes it will mean large in some topological sense. The notations which will be used for various measure theoretic and topological notions referred to throughout this article will now be given. It is assumed the reader is somewhat familiar with the definitions. They can be found in [23], [25], [26] and elsewhere, of course.

Some of the measure theoretic classes of sets which will be referred to are the following. L will denote the collection of all Lebesgue measurable subsets of $[0,1]$. $L_0 = \{$subsets of [0,1] of measure zero$\}$. non-$L_0 = \{$subsets of $[0,1]$ of positive outer measure$\}$. $L \setminus L_0 = \{$subsets of $[0,1]$ of positive measure$\}$. co-$L_0 = \{$subsets of $[0,1]$ of full measure$\}$.

Some of the topological classes of sets which will be referred to are the following. $ND = \{$nowhere dense subsets of $[0,1]\}$. $FC = \{$first category subsets of $[0,1]\}$. non-$FC = \{$second category subsets of $[0,1]\}$. co-$FC = \{$residual subsets of $[0,1]\}$. B_w denotes the collection of all subsets of $[0,1]$ which have the Baire property (in the wide sense) (i.e. the sets which have a first category symmetric difference with some Borel set). $B_w \setminus FC = \{$second category sets with Baire property$\}$.

The analogies that are customarily considered to exist between these measure theoretic notions and the corresponding topological notions will be embraced throughout this article. These analogies are listed in the following table.

SOME ANALOGIES:

	Topological	Measure Theoretic
Basic σ-algebras:	B_w	L
Very Small:	ND	
Small (σ-ideals):	FC	L_0
Large:	non-FC	non-L_0
Larger:	$B_w \setminus FC$	$L \setminus L_0$
Largest:	co-FC	co-L_0

3 The Classical Theorems

Consider the following classes of functions $f : [0,1] \to \mathbb{R}$ (C denotes the class of continuous functions).

$$\begin{array}{ccccc} & \nearrow & L\text{-measurable} & \searrow & \\ C & & & & \text{arbitrary} \\ & \searrow & B_w\text{-measurable} & \nearrow & \end{array}$$

The best known restriction theorem in real analysis is Lusin's Theorem about continuous restrictions of Lebesgue measurable functions, which can be found in almost any first year graduate analysis text.

(Lusin's First) Theorem 1. *For every L-measurable $f : [0,1] \to \mathbb{R}$, there exists an $M \in L \setminus L_0$ such that $f|M \in C(rel\ M)$.*

This theorem was stated by Lusin in 1912 [30], but it was known earlier by Lebesgue [28], Borel [4], and Vitali [45]. The current author thanks John Morgan III for providing him with these references. The theorem is usually stated in stronger "controlled intersection theorem" form in most text books, and it is pointed out that you can make the measure of the set M be as close to 1 as is desired. However, one cannot choose $M \in$ co-L_0 in Lusin's Theorem (this fact provides for the current author's favorite homework problem to assign in first year graduate analysis courses). On the other hand, Lusin showed in 1916 that one could obtain a set M of full measure if one is willing to relax a bit on the niceness of the restricted function.

(Lusin's Second) Theorem 2. *For every L-measurable $f : [0,1] \to \mathbb{R}$, there exists an $F \in CaeD^1$ and an $M \in$ co-L_0 such that $f|M = F'|M$.*

$CaeD^1$ denotes the class of continuous functions which are almost everywhere differentiable. This theorem might be referred to as "Lusin's Derivative Restriction Theorem".

The second most well-known restriction theorem in real analysis is probably Blumberg's Theorem about continuous restrictions of arbitrary functions, which was proved [3] in 1922.

(Blumberg's) Theorem 3. *For every $f : [0,1] \to \mathbb{R}$, there exists $M \subseteq [0,1]$, M dense in [0,1] such that $f|M \in C(rel\ M)$.*

It is clear from reading proofs of this theorem that the set M which is constructed in the proof, while dense in $[0,1]$, is nevertheless countable. Is was known in the 1920's that the set M could not be made to have cardinality c

(cardinality of the continuum), because of the "Sierpinski-Zygmund function" [43] $f : [0,1] \to \mathbb{R}$ which has no continuous restriction to any set of cardinality c.

Possibly the third best known restriction theorem dating to the 1920's is the following theorem proved by Nikodym [37] in 1929.

(Nikodym's) Theorem 4. *A function $f : [0,1] \to \mathbb{R}$ is B_w-measurable if and only if there exists $M \in$ co-FC such that $f|M \in C(rel\ M)$.*

The class of functions in the conclusion of Theorem 4, called the "functions with the property of Baire", was studied earlier. Nikodym defined the collection B_w and showed that it was a σ-algebra which could be used to characterize those functions. Nikodym's theorem was proved in the general separable metric setting by Kuratowski [24] in 1930.

4 The Continuous Case

In this section, the restriction and intersections theorems concerning the following collections of functions are considered.

$$A \to C^\infty \to \cdots \to C^2 \to D^2 \to C^1 \to D^1 \to \text{“}D^1\text{”} \to C$$

A denotes the class of real analytic functions. The classes C^∞, C^n, and D^n denote the infinitely differentiable, n-times continuously differentiable, and n-times differentiable functions, respectively. The class "D^1" is the collection of continuous functions which are differentiable in the "extended" sense (i.e. $+\infty$ and $-\infty$ are allowed values of $f'(x)$). To appreciate the significance of the theorems to be discussed below, one must remember how badly non-differentiable functions $f \in C$ can be. The most common continuous nowhere differentiable function one finds discussed in most real analysis texts would be the function described by Weierstrass in 1875.

$$f(x) = \sum_{n=0}^{\infty} a^n cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer, and $ab > 1+(\frac{3}{2})\pi$.

Other texts would include a description of the function described by van der Waerden in 1930.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} g(12^n x)$$

where g is the "distance to nearest integer" function.

Both of the above functions are nowhere "D^1" but they do have 1-sided "D^1" points. Indeed, this behavior is "typical" of functions $f \in C$, in the sense that the set of functions which are not like this forms a first category subset of C, topologized with the sup norm. Examples due to Besicovich (1925) and Morse (1938) are worse than the Weierstrass and van der Waerden functions in that they are nowhere 1-sided "D^1". See chapter 7 of the book by Jeffery [23] for an expanded discussion of continuous nowhere differentiable functions.

Nevertheless, continuous functions do have nice restrictions to perfect sets. A set is perfect if it is closed and has no isolated points (e.g. closed intervals, Cantor sets, and closed unions of such sets). Such sets are considered to be large in the sense that they must have cardinality c. The following was probably known to many mathematicians in the 1930's or 40's.

(Folk) Theorem 5. *For every $f \in C$, there exists a perfect $P \subseteq [0,1]$ such that $f|P = g|P$ for some $g \in D^1$.*

The current author is unable to provide a reference to this result, but it would have followed from Lebesgue's 1904 Differentiability Theorem [29], Jarník's 1923 Differentiable Function Extension Theorem [21], and known results concerning nowhere monotone functions such as those given in 1940 by Minakshisundarum [35] (also see [40]). It would also have been known at that time that you can't choose the set P in the above theorem to be in $L \setminus L_0$ because of pathological nondifferentiable continuous functions described by Jarník [22] in 1934.

At about that time (early 1940's), Ulam asked [Scottish Book Problem 17.1] (see [44]) whether every continuous function agrees with some real analytic function on some uncountable set. Zahorski showed in 1947 [48] that the answer is no because there exists a C^∞ function which has only finite intersection with every real analytic function. Zahorski raised the natural question concerning intersections of continuous functions with functions in smoother classes, and this became known as the

Ulam-Zahorski Problem. *If $f \in C$, does there necessarily exists a perfect $P \subseteq [0,1]$ and a $g \in C^\infty$ (or C^n or D^n) such that $f|P = g|P$?*

The fact that Zahorski stated the question as being open in the D^1 case would indicate that he was not aware of Theorem 5 stated above.

A theorem which does not provide a solution to the Ulam-Zahorski Problem but which was a step in the positive direction was proved by Bruckner, Ceder, and Weiss [14] in 1969.

Theorem 6. *For every $f \in C$ and every perfect $P \subseteq [0,1]$, there exists a perfect $Q \subseteq P$ such that $f|Q \in$ "D^1"(rel Q).*

This theorem would be referred to as the $C -$ "D^1" Controlled Restriction Theorem. The quotation marks are necessary in the notation for the differentiability class because the set P could be of measure zero, in which case it is possible that $f|P$ could already have all derivative values $= +\infty$. A nice alternative proof of the above theorem was given by Morayne [36] in 1985.

The following theorem includes a contribution to the solution (in the positive direction) to the Ulam-Zahorski Problem. It is the current author's favorite theorem.

Theorem 7. *For every perfect $P \in L \setminus L_0$ and every $f \in C$, there exists a perfect $Q \subseteq P$ such that*

(i) $f|Q \in C^\infty(rel\ Q)$, and

(ii) $f|Q = g|Q$ for some $g \in C^1$.

Theorem 7 with conclusion (*i*) was proved by Laczkovich [27] in 1984 and would be called the $C - C^\infty$ Controlled Restriction Theorem. Theorem 7 with conclusion (*ii*) was proved by Agronsky, Bruckner, Laczkovich, and Preiss [1] in 1985 and would be called the $C - C^1$ Controlled Intersection Theorem. The latter result represents a step in the positive direction toward the complete solution to the Ulam-Zahorski Problem. The proof of this part of the theorem involves restricting down to a perfect set Q in such a way that the 1934 Whitney Extension Theorem [46] is satisfied so that one can conclude that $f|Q$ is extendable to some $g \in C^1$. The authors show by example in [1] that the function g of part (*ii*) cannot be chosen to be $\in C^2$ (or even D^2). However, this still leaves open whether or not one can go further in the positive direction in the solution to the Ulam-Zahorski Problem. Indeed, the authors raised the following question: *if $f \in C$, does there exist a perfect $P \subseteq [0,1]$ and a $g \in C^2$ such that $f|P = g|P$?* The answer to this question was provided by Olevskiĭ in 1994, and his paper is discussed in the Section 7.

The nature of part (*ii*) of Theorem 7 suggested to the current author that there should be a similar conclusion which could be added to Theorem 6. Consider the following classes of functions.

$$\begin{tikzcd}
& \text{“}C^1\text{”} \arrow[dr] & & \\
C^1 \arrow[ur] \arrow[dr] & & \text{“}D^1\text{”} \arrow[r] & C \\
& D^1 \arrow[ur] & &
\end{tikzcd}$$

where the class "C^1" is the collection of those functions $f \in$ "D^1" for which the derivative f', considered to be an extended real valued function, is continuous in the extended sense. The following C – "C^1" Controlled Intersection Theorem was proved in 1990 [10].

Theorem 8. *For every $f \in C$ and every perfect $P \subseteq [0,1]$, there exists a perfect $Q \subseteq P$ and a $g \in$ "C^1" such that $f|Q = g|Q$.*

The proof of Theorem 8 required first proving a new "C^1" variant of the Whitney Extension Theorem in the real case.

It is not difficult to show (see [12]) that a partial converse to Theorem 7 (*ii*) holds in that *if* $M \in L_0$*, there is an* $f \in C$ *such that* $\{x \in M \; : \; f(x) = g(x)\}$ *is countable for every* $g \in C^1$. The following is suggested as an open problem.

Problem 1. *Is it the case that for every set* M $\in$*non-*L_0 *and every* $f \in C$*, there exists an uncountable* $N \subseteq M$ *such that*

(i) $f|N \in C^\infty(rel\ N)$*, or*

(ii) $f|N = g|N$ *for some* $g \in C^1$*?*

5 Variations on Blumberg's Theorem

Consider the relationship between Lusin's First and Second Theorems. He was able to obtain nice restrictions of Lebesgue measurable functions to sets of full measure by relaxing the "niceness" of the restriction from continuity to the condition of being a derivative in some sense. In 1971, the current author obtained a variant of Blumberg's Theorem which was of a similar flavor. As was pointed out earlier, one cannot choose the set M of Blumberg's Theorem to be of cardinality c, but the following theorem was proved in [6].

Theorem 9. *For every* $f : [0,1] \to \mathbb{R}$ *there exists a set* $N \subseteq [0,1]$*,* N *c-dense in [0,1] such that* $f|N \in PWD(rel\ N)$.

N is c-dense in [0,1] if every subinterval of [0,1] intersects N in a set of cardinality c. The class, PWD, is the collection of functions which are "pointwise discontinuous" (i.e. continuous at the elements of dense subsets of their domains). The relationship between this class, the class, B^1, of Baire-1 functions (pointwise limits of sequences of continuous functions), and the class, Δ, of derivatives is indicated in the following diagram, which holds for functions with domains [0,1].

$$C \to \Delta \to B^1 \to PWD \to arbitrary$$

Of course, it would have been preferable to have $f|N$ be in $\Delta(rel\ N)$ in the conclusion of Theorem 9 so that it would me more like Lusin's Second Theorem. However, it is not possible to obtain this. The reasons it would be impossible to obtain $f|N \in B^1(rel\ N)$ were discussed in [7] and the following relationship

$$\Delta(rel\ N) \to B^1(rel\ N)$$

for sets N which have no isolated points was established in [41].

It followed from a general version of Blumberg's theorem proved by Bradford and Goffman in 1960 [5] that the set M of Blumberg's Theorem 3 could be chosen to fall inside any preassigned set K which is "categorically dense" in [0,1] (i.e. every subinterval of [0,1] intersects K in a non-FC set), thus yielding a controlled variant of Blumberg's Theorem. It was actually shown in [6] that the set N of Theorem 9 could be forced to fall within any preassigned set K which satisfies a more complicated density property which involves Baire category and so-called "Lusin sets" [31] (uncountable sets which have only countable ND subsets).

Whereas it was known that the set M of Blumberg's Theorem could not be chosen to have cardinality c, it was only recently shown by S. Baldwin [2] in 1990 that it is *consistent* with the axioms of set theory that the set M in Blumberg's Theorem could always be chosen to be *uncountably* dense in [0,1].

The C – "D^1" Controlled Restriction Theorem 6 suggested the following variant of Blumberg's Theorem, which was proved by J. Ceder [16] in 1969.

Theorem 10. *For every function $f : [0,1] \to \mathbb{R}$ and every uncountable $M \subseteq [0,1]$, there exists $N \subseteq M$, N bilaterally-dense-in-inself such that $f|N \in$ "D^1"$(rel\ N)$.*

A set N is bilaterally-dense-in-itself if every point of N is a 2-sided limit point of N. A small error in the proof of Theorem 10 was corrected in papers by Holický, [20] and the current author [7]. The latter paper also contains a strengthened version of Theorem 10 in which the set N is forced to be "bilaterally-c-dense-in-itself" and the restriction $f|N$ is "D^1"$(rel\ N)$ at each element of a dense subset of N (the requirement on M is stronger than just uncountability).

The statement of Theorem 8 would suggest the possibility that there should be a Controlled "C^1" Intersection variant of Ceder's Theorem 10. This is an **open problem**, which is formulated as follows.

Problem 2. *Is it the case that for every $f : [0,1] \to \mathbb{R}$ and every uncountable $M \subseteq [0,1]$, there exists $N \subseteq M$, N bilaterally-dense-in-inself such that $f|N = g|N$ for some $g \in$ "C^1"?*

The statement of Theorem 7 would suggest the possibility that there should be a Controlled C^∞ Restriction variant and a Controlled C^1 Intersection variant of Blumberg's Theorem.

Problem 3. *Is it the case that for every $f : [0,1] \to \mathbb{R}$ and every non-L_0 $M \subseteq [0,1]$, there exists $N \subseteq M$, N bilaterally-dense-in-inself such that*

(i) $f|N \in C^\infty(rel\ N)$ and

(ii) $f|N = g|N$ for some $g \in C^1$?

Additional variations on Blumberg's Theorem can be found in [8] and the survey article [9] is the text of a talk the current author gave on the topic in 1983 and contains 35 references on the subject.

6 Measurable Functions

A combination of Lusin's First and Second Theorems and Theorem 7 easily gets the following.

Theorem 11. *For every L-measurable* $f : [0,1] \to \mathbb{R}$,

(1) there exists a perfect $P \in L \setminus L_0$ *such that* $f|P \in C(rel\ P)$,

(2) there exists a co-L_0 M *and an* $F \in CaeD^1$ *such that* $f|M = F'|M$, *and*

(3) there exists a perfect $P \subseteq [0,1]$ *such that*

  (i) $f|P \in C^\infty(rel\ P)$, *and*

  (ii) $f|P = g|P$ *for some* $g \in C^1$.

Papers [11] and [13] by the current author and K. Prikry contain similar results concerning continuous-, derivative-, and differentiable-restriction theorems and related examples for the B_w-measurable functions as well as the functions which are measurable with the respect to the following σ-algebras. B denotes the class of Borel sets. B_r denotes the collection of sets M which have the "Baire property in the restricted sense" [26] (i.e. $M \cap P \in B_w(rel\ P)$ for every perfect $P \subseteq [0,1]$). U denotes the collection of sets M which are "universally measurable" [34] (i.e. M is measurable with respect to the completion of every Borel measure on [0,1]). (s) denotes the collection of sets M which are "Marczewski measurable" [33] [42] (i.e. every perfect $P \subseteq [0,1]$ contains a perfect subset $Q \subseteq P$ which is either a subset of M or misses M). The corresponding classes of measurable functions are related as follows.

$$\begin{tikzcd}
 & & L\text{-measurable} \\
 & U\text{-measurable} \arrow[ur] \arrow[dr] & \\
B\text{-measurable} \arrow[ur] \arrow[dr] & & (s)\text{-measurable} \\
 & B_r\text{-measurable} \arrow[ur] \arrow[dr] & \\
 & & B_w\text{-measurable}
\end{tikzcd}$$

The main theorems given in [11] and [13] are summarized in the following.

Theorem 12. *For every B_w-measurable $f : [0,1] \to \mathbb{R}$,*

(1) there exists a co-FC set M such that $f|P \in C(rel\ M)$,

(2) there exists a co-FC set M and an $F \in D^1$ such that $f|M = F'|M$, and

(3) there exists a perfect $P \subseteq [0,1]$ and a $g \in$ "C^1" such that $f|M = g|M$.

Theorem 13. *For every (s)-measurable $f : [0,1] \to \mathbb{R}$,*

(1) there exists a perfectly dense set $M \subseteq [0,1]$ such that $f|P \in C(rel\ P)$,

(2) there exists a perfectly dense set $M \subseteq [0,1]$ and an $F \in D^1$ such that $f|M = F'|M$, and

(3) there exists a perfect $P \subseteq [0,1]$ and a $g \in$ "C^1" such that $f|M = g|M$.

A set $M \subseteq [0,1]$ is perfectly dense in [0,1] if every subinterval of [0,1] contains a perfect subset of M, so this condition is stronger than the condition of being c-dense in [0,1]. Theorem 13 (*1*) appeared in [13] and provided a positive response to a question raised in [17].

Problem 4. *There are numerous examples given in [11] and [13] which show (sometimes under assumption of the Continuum Hypothesis) that parts (1) and (2) of Theorems 11, 12, and 13 cannot be strengthened in various ways, even for U-measurable or B_r-measurable functions. It would be nice to know whether these examples can be constructed without assumption of the Continuum Hypothesis.*

7 Lipschitz and Hölder Class Functions

The function classes to be taken up in this section are the following.

$$\begin{tikzcd}
C^{(2+\alpha)} \arrow[dr] & & C^{(1+\alpha)} \arrow[dr] & & Lip^\alpha \arrow[dr] & \\
& C^2 \arrow[ur] \arrow[dr] & & C^1 \arrow[ur] \arrow[dr] & & C \\
D^3 \arrow[ur] & & D^2 \arrow[ur] & & D^1 \arrow[ur] &
\end{tikzcd}$$

where $1 \geq \alpha > 0$.

$f \in Lip^\alpha$ if there exists a $K > 0$ such that $|f(y) - f(x)| \leq K|y-x|^\alpha$ for all $x, y \in [0,1]$. The Lip^α classes get smaller with increasing α, Lip^1 being the smallest of these classes. $C^{(1+\alpha)}$ denotes the class of all $f \in C^1$ for which f' is in Lip^α. $C^{(2+\alpha)}$ denotes the class of all $f \in C^2$ for which f'' is in Lip^α.

The "final solution" of the Ulam-Zahorski Problem was published in 1994 by A. Olevskiĭ [39]. This beautiful paper contains a number of extremely interesting results. Olevskiĭ first describes an example of a function $f \in Lip^1$ such that for all $\alpha > 0$ and every $g \in C^{(1+\alpha)}$, $f \cap g$ is countable. Olevskiĭ's example is described as follows.

$$f(x) = \int_0^x \sum_{n=1}^{\infty} b^n r(v_n t) dt$$

where $r =$ the Rademacher function, defined by

$$r(x) = \begin{cases} -1 & \text{for } 0 < x < .5, \\ 1 & \text{for } .5 < x < 1, \end{cases}$$

r extended so as to be periodic on $\mathbb{R}$ with period 1, $0 < b < 1$, and v_n increases rapidly toward $+\infty$. Olevskiĭ actually announced this and other results described below several years earlier than the proofs were published (see [38]). The current author worked out his own argument that the example given above satisfied the stated requirements. In the process of working this out it was noticed that with $b = .1$ and $v_n = n^n!$, for example, $f \cap h$ is uncountable for some $h \in$ "C^2" (see [11]). This suggested the following possibility.

Problem 5. *Is it possibly the case that every $f \in C$ (or Lip^1) has uncountable intersection with some $h \in$ "C^2" (or "D^2")?*

The current author convinced himself that the above example also shows that functions in C cannot be expected to have uncountable intersections with functions in class D^2, but no proof of this additional observation has been published. On the other hand, Olevskiĭ went on to prove the following positive result related to the Ulam-Zahorski Problem.

Theorem 14. *For every $f \in C^1$, there exists a perfect $P \subseteq [0,1]$ such that $f|P = g|P$ for some $g \in C^2$.*

Then Olevskiĭ showed that you cannot go further up the C^n chain with similar positive results because of the following.

Example 15. *There exists $f \in C^2$ such that for every $\alpha > 0$ and every $g \in C^{(2+\alpha)}$, $f \cap g$ is countable.*

Upon reading Theorem 14, the current author wondered "Why didn't Olevskiĭ prove a $C^1 - C^2$ Controlled Intersection Theorem instead of just the Intersection Theorem?" Possibly it could be shown, as was the case in Theorem 7 (*ii*), that for every $M \in L \setminus L_0$ the perfect set P in Theorem 14 could be chosen to lie inside M. This was not possible and an example to show so was already given in [1]. Then perhaps one could accomplish this result for every co-L_0 set M. A counterexample to this conjecture was constructed and is given in [12]. If large measure doesn't provide the control, perhaps one could accomplish the result if M is co-FC. This also turned out to fail and a counterexample is given in [12]. A condition was finally found which suffices and Olevskiĭ's proof can be modified to yield the following $C^1 - C^2$ Controlled Intersection Theorem [12].

Theorem 16. *For every $f \in C^1$, and every $M \in B_w \cap L$ which is simultaneously measure-dense and residual in some subinterval of [0,1], there exists a perfect $P \subseteq M$ such that $f|P = g|P$ for some $g \in C^2$.*

M is measure-dense in an interval $[a, b]$ if every subinterval of $[a, b]$ intersects M in a set of positive measure. The examples mentioned above show that one can't weaken the requirements on M in Theorem 16 much and still obtain the conclusion. It is conjectured that the condition on M is exactly the right one in the sense that there will be a positive solution to the following.

Problem 6. *Is it the case that if $M \in B_w \cap L$ is not simultaneously measure dense and residual in some subinterval of [0,1], then there will exist an $f \in C^1$ such that for every $g \in C^2, \{x \in M | f(x) = g(x)\}$ is countable?*

There is a proof of Theorem 7 (*ii*) which calls upon a theorem due to Federer [18] (1944) which might be called the "$Lip^1 - C^1$ Controlled Intersection Theorem". The proofs of Theorems 14 and 16 rely directly upon a generalization of Federer's theorem proved by Whitney [47] in 1951 and which might be called the "$C^{(2)} - C^2$ Controlled Intersection Theorem". Those two theorems are the following.

Theorem 17. *For every $f \in D^1 \cup Lip^1$ and every $M \in L \setminus L_0$, there exists a $g \in C^1$ such that $\{x \in M \ : f(x) = g(x)\} \ \in \ L \setminus L_0$.*

Theorem 18. *For every $f \in D^2 \cup C^{(2)}$ and every $M \in L \setminus L_0$, there exists a $g \in C^2$ such that $\{x \in M \ : f(x) = g(x)\} \ \in \ L \setminus L_0$.*

In [12], the current author proved the following topological variants of these two theorems.

Theorem 19. *For every $f \in D^1 \cup Lip^1$ and every $M \in B_w \setminus FC$, there exists $g \in C^1 \ \ni \ \{x \in M \ : f(x) = g(x)\}$ is uncountable.*

Theorem 20. *For every $f \in D^2 \cup C^{(2)}$ and every $M \in B_w \setminus FC$, there exists $g \in C^2 \ \ni \ \{x \in M \ : f(x) = g(x)\}$ is uncountable.*

It is the case that one can't get $\{x \in M \ : f(x) = g(x)\} \in B_w \setminus FC$ in the conclusion of Theorems 19 and 20. Federer's Theorem 17 was actually proved for functions which are continuous and almost everywhere pointwise Lip^1 rather than just for the functions in $D^1 \cup Lip^1$, but one cannot prove Theorem 19 for this larger class. Similarly, Whitney's Theorem 20 was proved for functions whose derivatives are continuous and almost everywhere pointwise Lip^1 rather than just for the functions in $D^2 \cup C^{(2)}$, but one cannot prove Theorem 20 for this larger class.

A combination of Theorems 17 and 19 yields the following fact: *If* $M \in (L \setminus L_0) \cup (B_w \setminus FC)$, *then for every* $f \in D^1 \cup Lip^1$, *there exists a* $g \in C^1 \ni \{x \in M : f(x) = g(x)\}$ *is uncountable.*

Problem 7. *Is it the case that if* $M \in L_0 \cap FC$, *then there exists an* $f \in D^1 \cap Lip^1$ *such that for every* $g \in C^1$, $\{x \in M : f(x) = g(x)\}$ *is countable?*

Consideration of a similar combination of Theorems 18 and 20 leads to following question.

Problem 8. *Is it the case that if* $M \in L_0 \cap FC$, *then there exists an* $f \in D^2 \cap C^{(2)}$ *such that for every* $g \in C^2$, $\{x \in M : f(x) = g(x)\}$ *is countable?*

Of course, if a positive solution is given for Problem 7, Problem 8 becomes moot.

Finally, it should be pointed out that Whitney actually generalized Federer's $Lip^1 - C^1$ Controlled Intersection Theorem to a $C^{(n)} - C^n$ Controlled Intersection Theorem which is true for all n (see [19]). It is natural to ask if Theorems 19 and 20 can be extended to the general case.

Problem 9. *Is it the case that if* $M \in B_w \setminus FC$, *then for every* $f \in D^n \cup C^{(n)}$, *there exists a* $g \in C^n$ *such that* $\{x \in M : f(x) = g(x)\}$ *is uncountable?*

Final Remark. Bruckner and Jones have recently published an expository paper [15] on the subject of this article. However, the emphasis in the two papers is different. This would be indicated by the fact that, while the two papers do have 9 common references, the symmetric difference of the two sets of references is of cardinality 76.

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