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THIN SETS OF HARMONIC ANALYSIS AND INFINITE COMBINATORICS

Abstract

This is a survey paper on some classical trigonometric families of thin sets (Dirichlet sets, weak Dirichlet sets, N-sets, N_0-sets, A-sets, U-sets, and two recently introduced families of B-sets and of B_0-sets), the relationships between them, and basic closure properties of these families, presented as complete answers to ten questions. However, a large part of the paper is devoted to presentation of new results. In addition, we tried to give an overview of the best known estimates for cardinal characteristics for these families and for the families of particularly "permitted" sets, using small uncountable cardinals recently studied in infinite combinatorics. Almost all results are accompanied by brief notes on the investigations preceding them. Finally, we study properties of families of thin sets related to the Rademacher and Walsh orthogonal systems of functions. Some of these families are studied for the first time.

Key Words: Trigonometric series, uniqueness, absolute convergence, thin sets, permitted sets, small uncountable cardinals, cardinal characteristics, γ-set, consistency, Rademacher system, Walsh system.

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Contents

The paper is intended to be a survey of the topics announced in the title, starting with the very beginning of the research area, giving a brief survey of its development, and including recent unpublished results. The first section contains a brief survey of the results which started the investigation of thin sets. Sections 3–5 summarize the recent tools and results of set theory which will be used in our considerations. Sections 6–9 present rather classical results concerning thin sets of trigonometric series, in spite of the fact that some of them are quite recent. Section 10 is an interlude showing that we did not forget any reasonable type of thin set. In Section 11 we give complete answers to a set of ten basic questions about classical families of thin sets. Sections 12–16 and 18 are devoted to the computation of cardinal characteristics of these families of thin sets and present some related results. They contain both recent and new results. Actually, the results presented in Sections 12 (12.2–12.6), 13 (13.3–13.5), 14, 15, 16 (16.1–16.4) and 18 (18.3–18.7) are published for the first time. Some of them were presented at the conferences "Problems in Real Analysis" in Lodź, July, 1994 [BL2] and "Summer School on Real Functions Theory" in Liptovský Ján, September, 1994.

We tried to ascribe each result to its author or authors by indicating the corresponding bibliographic source preceding its formulation (either as a quotation or as a theorem).

1 Brief history

In 1807, Joseph Fourier submitted a basic paper on heat conduction to the Academy of Sciences of Paris. The paper was judged by J. L. Lagrange, P. S. Laplace and A. M. Legendre and was rejected. In 1811, Fourier submitted a revised paper for a competition of the Academy. He won the prize, but the paper was not published at that time because of a lack of rigor. The first part of this revised paper was incorporated into one of the classics of mathematics, *Théorie analytique de la chaleur* [Fou] (see [Kli] for more details). In this book (and already in the 1807 paper), using some geometrical reasoning, Fourier concluded that every function could be represented as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi n x + b_n \sin 2\pi n x), \tag{1.1}$$

for $x \in (0,1)$. He also claimed that this series is convergent for any function f, whether or not one can assign an analytic expression to f and whether or not f follows any regular law. Of course, that is not true. Nor did the mathematical authorities of that time believe it—that was one of the reasons Lagrange rejected the paper. However, mathematicians of the 19th century started to study the possibility of such a representation (1.1). Let us remark that B. Riemann introduced his notion of integral while studying Fourier series, and many important results by U. Dini, P. G. Lejeune Dirichlet, and others are connected with this topic.

A series (1.1), where[1] $a_n, b_n \in \mathbb{R}$, $n = 0, 1, \ldots$, is called a ***trigonometric series***.

In 1870 G. Cantor [Ca1] proved the first uniqueness result:

> *If the trigonometric series* (1.1) *converges to* 0 *for all* $x \in [0,1]$, *then all* a_n, b_n, $n = 0, 1, \ldots$, *are equal to* 0.

Later, Cantor realized that the theorem remains true when the words "for all" are replaced by "for all but finitely many". Finally, Cantor extended the theorem to the case of a countable set of finite Cantor-Bendixson rank of exceptions, introducing in [Ca3] the notions of "Wertmenge" and "Punktmenge". This paper actually started the development of set theory.

Cantor's result was generalized by W. H. Young [You]:

Young Theorem 1.1 *If the trigonometric series* (1.1) *converges to* 0 *for every* $x \in [0,1]$ *outside a countable set, then all* a_n, b_n, $n = 0, 1, \ldots$, *are equal to* 0.

In 1871 Cantor [Ca2] proved for a closed interval, and H. Lebesgue proved in the general case (for the proof see [Ba2, KL, Zy1]):

Cantor-Lebesgue Theorem 1.2 *If the trigonometric series* (1.1) *converges on a set of positive Lebesgue measure, or even if*

$$\lim_{n\to\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx) = 0$$

on a set of positive Lebesgue measure, then

$$\lim_{n\to\infty} (|a_n| + |b_n|) = 0.$$

In 1912 A. Denjoy [Den] and N. N. Luzin [Lu1] independently proved the following

Denjoy-Luzin Theorem 1.3 *If the trigonometric series* (1.1) *converges absolutely on a set of positive Lebesgue measure, then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty,$$

i.e. the trigonometric series (1.1) *converges absolutely everywhere.*

[1] For simplicity, in the whole paper, we assume $b_0 = 0$.

In 1915 Luzin [Lu2] showed

Luzin Theorem 1.4 *If the trigonometric series* (1.1) *converges absolutely on a non-meager set then*

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty.$$

An obvious question to ask is: are the results presented in these theorems in some sense the best possible? Does the conclusion of Young's theorem hold true for some uncountable set? Do the conclusions of the Cantor-Lebesgue and Denjoy-Luzin theorems hold true for some set of measure zero? Does the conclusion of Luzin's theorem hold true for some meager set? Is there a convenient characterization of the sets for which the assumption of convergence in each of the above four theorems is sufficient for the conclusion?

2 Notations and terminology

We work in Zermelo-Fraenkel axiomatic set theory with the axiom of choice, ZFC; see e.g. [Jech]. We believe that this theory is consistent; i.e. one cannot prove in ZFC both a sentence and its negation. Then for any sentence φ we have three mutually exclusive possibilities: (1) φ can be proved in ZFC, (2) its negation $\neg\varphi$ can be proved in ZFC, or (3) neither φ nor its negation $\neg\varphi$ can be proved in ZFC. If we want to show that some φ can be proved in ZFC, we simply write the proof or give an adequate reference for such a proof. If we want to show that φ cannot be proved in ZFC, usually we construct a model of ZFC in which $\neg\varphi$ holds true. As is customary in contemporary mathematics, by saying "holds true" we mean "can be proved in ZFC".

We use standard set-theoretic terminology and notations such as those of [Jech, Vau]. If φ is a formula and X is a set then the set of all elements of X satisfying the formula φ will be denoted by

$$\{x \in X : \varphi(x)\},$$

and the set of all subsets of X satisfying the formula φ will be denoted by

$$\{x \subseteq X : \varphi(x)\}.$$

Similarly, if f is a function defined for all x which satisfy φ, we denote the set (if it does exist) of those $f(x)$'s by

$$\{f(x) : \varphi(x)\}.$$

The set of all natural numbers $0, 1, 2, \ldots$ will be denoted by ω. If $n \in \omega$, then we identify n with the set of all smaller natural numbers, i.e.

$$n = \{i \in \omega : i < n\}.$$

The set of all functions defined on the set X with values in the set Y is denoted by ${}^X Y$. For example, ${}^\omega 2$ is the set of all infinite sequences of 0's and 1's. $|X|$ is the cardinality of the set X. In particular, $\aleph_0 = |\omega|$ and $\mathfrak{c} = |\mathbb{R}|$. The set X is said to be finite (countable) if $|X| < \aleph_0$ ($|X| \leq \aleph_0$).

Let us recall that a sequence of real-valued functions $\{f_n\}_{n=0}^\infty$ ***quasinormally converges*** to a function f on a set X if there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ converging to zero such that

$$(\forall x \in X)(\exists k)(\forall n > k)\, |f_n(x) - f(x)| \leq \varepsilon_n.$$

Quasinormal convergence[2] was introduced and studied in [BZ2, CL]. The main property of quasinormal convergence that we shall need is the following simple

Theorem 2.1 *If the sequence of real-valued functions $\{f_n(x)\}_{n=0}^\infty$ quasinormally converges to 0 on a set X, then there is a strictly increasing sequence $\{n_k\}_{k=0}^\infty$ of natural numbers such that the series $\sum_{k=0}^\infty f_{n_k}(x)$ absolutely converges on X.*

We denote by $||x||$ the distance of the real x to the nearest integer, i.e.

$$||x|| = \min\{\{x\}, 1 - \{x\}\},$$

where $\{x\}$ is the fractional part of the real x. One can easily see that

$$||x + y|| \leq ||x|| + ||y||,$$
$$2||x|| \leq |\sin \pi x| \leq \pi||x||$$

for any reals x, y. So, we can in our considerations mutually replace the functions $||x||$ and $|\sin \pi x|$.

We shall need a modification of the classical Dirichlet-Minkowski Theorem (which is a special case of the following theorem with $n_i = i$; see e.g. [Ba2, LP]).

Theorem 2.2 *Let $\{n_i\}_{i=0}^\infty$ be a strictly increasing sequence of natural numbers. For any $\varepsilon > 0$ and any reals $x_1, \ldots, x_k$, there are i, j such that $0 \leq i < j \leq (1/\varepsilon)^k$ and*

$$(2.1) \qquad ||(n_j - n_i)x_l|| < 2\varepsilon \qquad \text{for } l = 1, 2, \ldots, k.$$

[2] Á. Császár and M. Laczkovich call it equal convergence.

PROOF. We can assume that $\varepsilon < 1/2$. Let $m \in \omega$ be such that $\varepsilon \leq 1/m < 2\varepsilon$. We divide the k-dimensional cube $[0,1]^k$ into $t = m^k$ equal cubes of side $1/m$. By the pigeon-hole principle, from the $t+1$ elements

$$(\{n_i x_1\}, \ldots, \{n_i x_k\}), \quad i = 0, 1, \ldots, t$$

at least two are in the same cube; i.e., there are $i \neq j$ such that (2.1) holds true and $|j - i| \leq m^k \leq (1/\varepsilon)^k$.

□

For a subset A of $[0,1]$ and a real x, we denote the shift of A by

$$x + A = \{\{x + a\} : a \in A\}$$

and the expansion of A by

$$xA = \{\{xa\} : a \in A\}.$$

3 Small and thin sets

Let $\mathcal{F}$ be a family of subsets of a set X. A subfamily $\mathcal{G} \subseteq \mathcal{F}$ is called a ***basis*** of $\mathcal{F}$ iff

$$(\forall A \in \mathcal{F})(\exists B \in \mathcal{G})\, A \subseteq B.$$

If $\mathcal{F}$ is a family of subsets of a topological space, then we speak about a Borel basis, an F_σ basis, etc., if the basis $\mathcal{G}$ consists of Borel sets, F_σ sets, etc., respectively. From the family $\mathcal{F}$, we may construct a new family $\mathcal{F}_\sigma$ by

$$\mathcal{F}_\sigma = \{\bigcup_{n \in \omega} A_n : A_n \in \mathcal{F} \text{ for } n \in \omega\}.$$

The typical small subsets of the real line or the unit interval $[0,1]$ are the meager (= of the first Baire category) sets or the negligible sets (= sets of Lebesgue measure zero). Since we shall use them often, we denote

$$\mathcal{K} = \{A \subseteq [0,1] : A \text{ is meager}\},$$
$$\mathcal{L} = \{A \subseteq [0,1] : A \text{ is negligible}\}.$$

The families $\mathcal{K}$ and $\mathcal{L}$ have an F_σ basis and a G_δ basis, respectively.

Other small sets of real analysis are the porous sets. We assume that the reader is familiar with L. Zajíček's paper [Zaj], and we use its terminology (actually we need only three notions: porous, bilaterally porous and σ-porous). The family of porous subsets of $[0,1]$ will be denoted by $\mathcal{P}$. The family of σ-porous sets is $\mathcal{P}_\sigma$. Every σ-porous set is contained in a σ-porous $\mathrm{G}_{\delta\sigma}$ set; i.e., $\mathcal{P}_\sigma$ has a $\mathrm{G}_{\delta\sigma}$ basis, see e.g. [FH].

A. Rajchman [Raj] introduced the notion of an H-set (H in honour of G. H. Hardy and J. E. Littlewood, who considered this kind of set): a set A is called an ***H-set*** if there are an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ and $0 \leq a, b < 1$ such that

$$0 \leq \{n_k x - a\} \leq b, \quad \text{for } x \in A \text{ and } k = 0, 1, 2, \ldots.$$

One can easily see that an H-set is a nowhere dense set of measure zero. On the other hand, every H-set is contained in a perfect H-set. If $\mathcal{H}$ denotes the family of all H-sets, then $\mathcal{H}_\sigma$ denotes the family of all countable unions of H-sets. Thus we have

$$\mathcal{H}_\sigma \subseteq \mathcal{K} \cap \mathcal{L}.$$

N. K. Bary [Ba2] presents an unpublished result of I. I. Piatetskiĭ-Shapiro which implicitly contains (see also [Zaj])

Theorem 3.1 *Every H-set is (bilaterally) porous.*

Let us recall that a family $\mathcal{F}$ of subsets of a set X is an ***ideal*** on X if

a) $\emptyset \in \mathcal{F}$, $X \notin \mathcal{F}$,

b) if $A \in \mathcal{F}$, $B \subseteq A$, then $B \in \mathcal{F}$,

c) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

An ideal $\mathcal{F}$ is called a ***σ-ideal*** if

d) whenever $A_n \in \mathcal{F}$ for $n \in \omega$, then $\bigcup_{n\in\omega} A_n \in \mathcal{F}$.

It is well known that $\mathcal{K}$, $\mathcal{L}$, $\mathcal{H}_\sigma$, $\mathcal{P}_\sigma$ are σ-ideals.

We usually say that a set $A \subseteq X$ is small with respect to some ideal $\mathcal{F}$ on X if $A \in \mathcal{F}$.

A very important indication of the bigness of a set $A \subseteq [0,1]$ is whether or not it contains a perfect subset. However, this property is not preserved by intersection, and some perfect sets are small (meager, measure zero, porous). A set $A \subseteq [0,1]$ is called a ***Bernstein set*** if neither A nor $[0,1] \setminus A$ contain a perfect subset. It is well known that

the axiom of choice implies the existence of Bernstein sets.

The notion of a Bernstein set is not a notion of smallness in the above sense.

As we shall see, the families of exceptional sets considered in trigonometric series theory usually do not form ideals, although the sets contained in them are often (not always) small in the above-mentioned sense. With these families in mind, we define: a family $\mathcal{F}$ of subsets of a set X (we consider only the case

$X = [0,1]$) is called a ***family of thin sets*** if $\mathcal{F}$ satisfies conditions a) and b) of the above definition.

Let $\mathcal{F}$ be a family of thin subsets of X. Let $A, B \subseteq X$. According to J. Arbault [Arb], the set A is said to be ***$\mathcal{F}$-permitted for the set B*** iff $A \cup B \in \mathcal{F}$. The set A is ***$\mathcal{F}$-permitted*** iff it is $\mathcal{F}$-permitted for every $B \in \mathcal{F}$. We denote

$$\mathrm{Prm}(\mathcal{F}) = \{A \subseteq X : A \text{ is } \mathcal{F}\text{-permitted}\}.$$

The following simple facts are implicitly contained in [Arb]:

1) $\mathrm{Prm}(\mathcal{F})$ is an ideal,

2) $\mathrm{Prm}(\mathcal{F}) \subseteq \mathcal{F}$,

3) $\mathrm{Prm}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathcal{F}$ is an ideal.

4 Cardinal characteristics

Let $\mathcal{F}$ be a family of subsets of a set X. The cardinal characteristics of the family $\mathcal{F}$ are defined as follows:

$$\begin{aligned}
\mathrm{non}(\mathcal{F}) &= \min\{|A| : A \subseteq X \ \&\ A \notin \mathcal{F}\},\\
\mathrm{add}(\mathcal{F}) &= \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F} \ \&\ \bigcup \mathcal{G} \notin \mathcal{F}\},\\
\mathrm{cov}(\mathcal{F}) &= \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F} \ \&\ X \subseteq \bigcup \mathcal{G}\},\\
\mathrm{cof}(\mathcal{F}) &= \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F} \ \&\ \mathcal{G} \text{ is a basis of } \mathcal{F}\}.
\end{aligned}$$

If the family $\mathcal{F}$ contains all singletons, $X \notin \mathcal{F}$ and $\bigcup \mathcal{F} = X$, then one can easily show that $\mathrm{add}(\mathcal{F}) \le \mathrm{cov}(\mathcal{F}) \le \mathrm{cof}(\mathcal{F})$ and $\mathrm{add}(\mathcal{F}) \le \mathrm{non}(\mathcal{F}) \le \mathrm{cof}(\mathcal{F})$.

The other cardinal characteristics we will consider are defined using partially ordered sets without minimal elements. A subset A of a partially ordered set $P, \le$ is ***open*** if for every $x \in A$ and $y \in P$, if $x \ge y$ then $y \in A$. A subset A of a partially ordered set $P, \le$ is ***dense*** in P if for every $x \in P$ there exists a $y \le x$, $y \in A$. A subset G of P is called a ***filter*** if for any $x, y \in G$ there exists a $z \in G$ such that $z \le x$ and $z \le y$; and, for any $x \in G$ if $x \le z$, then $z \in G$. Two elements x, y of a partially ordered set $P, \le$ are called ***disjoint*** if there is no $z \in P$ such that $z \le x$ and $z \le y$. A partially ordered set $P, \le$ is said to be ***C.C.C.*** if every subset of P consisting of pairwise disjoint elements is countable.

We recall that $\mathfrak{m}$ is the least cardinal κ for which there exists a C.C.C. partially ordered set $P, \le$ and a family $\{A_\xi : \xi < \kappa\}$ of dense subsets of P such that there is no filter on P meeting every A_ξ, $\xi < \kappa$.

It is easy to show that $\aleph_0 < \mathfrak{m} \leq \mathfrak{c}$, see e.g. [Fr1]. The assumption $\mathfrak{m} = \mathfrak{c}$ is called Martin's Axiom.

The family of all infinite sets of natural numbers is denoted by

$$[\omega]^\omega = \{A \subseteq \omega : A \text{ infinite}\}.$$

If $L \in [\omega]^\omega$ we denote by $L(n)$ the n^{th} element of L (starting from 0); i.e., $L = \{L(n) : n \in \omega\}$ and $L(n) < L(n+1)$ for every $n \in \omega$. For $X, Y \subseteq \omega$ let $X \subseteq^* Y$ denote that the set $X \setminus Y$ is finite. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family of infinite sets of natural numbers. We say that an infinite set $B \subseteq \omega$ is a ***pseudo-intersection*** of the family $\mathcal{F}$ if $B \subseteq^* A$ for all $A \in \mathcal{F}$. The family $\mathcal{F}$ has the ***finite intersection property*, *f.i.p.*** if for any finite system $A_1, \dots, A_n \in \mathcal{F}$, the intersection $\bigcap_{i=1}^n A_i$ is infinite. A family $\mathcal{F}$ of infinite subsets of ω is called a ***tower*** if the partially ordered set $\mathcal{F}, {}^*\supseteq$ is well-ordered and has no infinite pseudo-intersection. If A, B are subsets of ω we say that B ***splits*** A if both $A \cap B$ and $A \setminus B$ are infinite.

Generally, we say that some property of natural numbers "eventually holds true" if it is true for all but finitely many natural numbers. For example, $A \subseteq^* B$ if the implication $n \in A \Rightarrow n \in B$ eventually holds true. The set ${}^\omega\omega$ of all infinite sequences of natural numbers is partially quasi-ordered by the eventual dominating relation

$$f \leq^* g \equiv (\exists k \in \omega)(\forall n \in \omega)(n \geq k \Rightarrow f(n) \leq g(n)).$$

We need the following small uncountable cardinals which are cardinal characteristics of the structure of $\mathcal{P}(\omega)$:

$\mathfrak{p}$ is the least size of a family $\mathcal{F} \subseteq [\omega]^\omega$ with *f.i.p.* such that $\mathcal{F}$ has no infinite pseudo-intersection,

$\mathfrak{t}$ is the least size of a tower,

$\mathfrak{s}$ is the least size of a splitting family, i.e. the least size of a family $\mathcal{F} \subseteq [\omega]^\omega$ such that every infinite subset of ω is split by some set from $\mathcal{F}$,

$\mathfrak{r}$ is the least size of a family $\mathcal{F} \subseteq [\omega]^\omega$ such that no infinite subset of ω splits every member of $\mathcal{F}$.

$\mathfrak{h}$ is the least size of a family of open dense subsets of $[\omega]^\omega, \subseteq^*$ such that its intersection is not dense, or equivalently, it is the least κ such that the Boolean algebra $[\omega]^\omega/\textit{finite}$ is not κ-distributive.

$\mathfrak{b}$ is the least size of an unbounded subfamily of ${}^\omega\omega, \leq^*$,

$\mathfrak{d}$ is the least size of an cofinal (dominating) subfamily of ${}^\omega\omega, \leq^*$.

For basic information see e.g. [vDw, Vau].

We shall need [Boo]

Booth Lemma 4.1 *A set X has cardinality smaller than $\mathfrak{s}$ if and only if the following holds true: if $\{f_n\}_{n=0}^\infty$ is a sequence of functions defined on the* *set X with values in $[0,1]$, then there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\{f_{n_k}(x)\}_{k=0}^{\infty}$ converges for every $x \in X$.*

PROOF. Suppose $|X| < \mathfrak{s}$ and $\{f_n\}_{n=0}^{\infty}$ is a sequence of functions defined on X with values in $[0,1]$. For $q \in [0,1] \cap \mathbb{Q}$ and $x \in X$, let

$$L_{q,x} = \{n \in \omega : f_n(x) \leq q\}.$$

Since the family $\mathcal{F} = \{L_{q,x} : q \in [0,1] \cap \mathbb{Q}, x \in X\}$ cannot be splitting, there exists an infinite set $K \subseteq \omega$ such that for every $L \in \mathcal{F}$ either $K \subseteq^* L$ or $K \subseteq^* \omega \setminus L$. Then for each $x \in X$,

$$\lim_{n \in K} f_n(x) = \inf\{q \in [0,1] \cap \mathbb{Q} : K \subseteq^* L_{q,x}\}.$$

Conversely, if $\mathcal{F}$ is a splitting family with $|\mathcal{F}| = \mathfrak{s}$, we define the functions $f_n : \mathcal{F} \longrightarrow [0,1]$ by

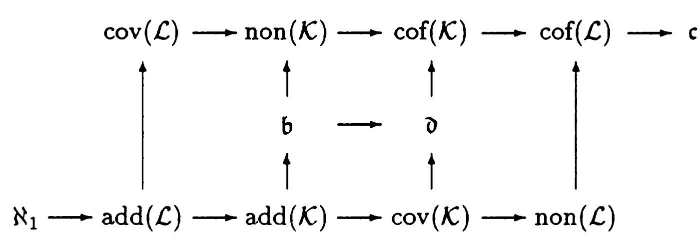
$$f_n(L) = \begin{cases} 1, & \text{if } n \in L, \\ 0, & \text{if } n \notin L. \end{cases}$$

It is easy to see that no subsequence of the sequence $\{f_n\}_{n=0}^{\infty}$ is convergent. □

Corollary 4.2 *Let $\{f_{n,0}\}_{n=0}^{\infty}, \ldots, \{f_{n,m}\}_{n=0}^{\infty}$ be sequences of functions defined on a set X with values in a closed interval $[a,b]$. If $|X| < \mathfrak{s}$, then there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that all sequences $\{f_{n_k,0}\}_{k=0}^{\infty}, \ldots, \{f_{n_k,m}\}_{k=0}^{\infty}$ converge on X.*

5 Diagrams

Now we present the main known relations between the cardinal characteristics of $\mathcal{L}$ and $\mathcal{K}$ and/or those of the structure of $\mathcal{P}(\omega)$. An arrow from a cardinal $\mathfrak{e}$ to a cardinal $\mathfrak{f}$ means that in ZFC the inequality $\mathfrak{e} \leq \mathfrak{f}$ is provable. We start with the Cichoń diagram [Fr2, Vau]:



It is worth to note that the equalities $\mathrm{add}(\mathcal{K}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{K})\}$ and $\mathrm{cof}(\mathcal{K}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{K})\}$ hold true. Moreover, it is known [BJS] that no equality can be proved; i.e., for every arrow $\mathfrak{e} \rightarrow \mathfrak{f}$ in this diagram, there exists a model of ZFC in which $\mathfrak{e} < \mathfrak{f}$.

The cardinal characteristics of the structure $\mathcal{P}(\omega)$ are related as indicated in the following diagram:

$$\begin{array}{ccccccccccc}
 & & & & & & \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} \\
 & & & & & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathrm{add}(\mathcal{L}) & & & & \mathfrak{h} & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{r} \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\aleph_1 & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{t} & \longrightarrow & \mathrm{add}(\mathcal{K}) & \longrightarrow & \mathrm{cov}(\mathcal{K})
\end{array}$$

It is known for almost all these inequalities that the equality cannot be proved. However, the situation is not as simple as in the Cichoń diagram, see e.g. [Vau]. We add some relations between cardinals from both diagrams:

$$\mathfrak{s} \le \mathrm{non}(\mathcal{K}), \quad \mathrm{cov}(\mathcal{K}) \le \mathfrak{r},$$
$$\mathfrak{s} \le \mathrm{non}(\mathcal{L}), \quad \mathrm{cov}(\mathcal{L}) \le \mathfrak{r}.$$

Let us remark that some symmetry appears in both diagrams. This is a consequence of two kinds of dualities: Rothberger duality for cardinal characteristics of $\mathcal{K}$ and $\mathcal{L}$ based on the decomposition of the unit interval as the union of a meager and a negligible set [Rot], and the duality between characteristics based on the inverse relation, see e.g. P. Vojtáš [Vo2].

6 Sets of uniqueness

For a recent and rather systematic treatment of sets of uniqueness, we recommend the book by A. Kechris and A. Louveau, [KL]. Following Cantor's results, we define: a set $A \subseteq [0,1]$ is said to be a ***set of uniqueness*** or ***U-set*** if every trigonometric series (1.1) converging to zero outside A is identically zero. The family of all U-sets will be denoted by $\mathcal{U}$.

So, Young's Theorem 1.1 can be formulated as "every countable set is a U-set". Using this result and the Alexandroff-Hausdorff Theorem, which asserts that every uncountable Borel set contains a perfect subset, (see e.g. [Jech, Theorem 94]) we can easily prove the following theorem: For a proof of assertions (1)–(5) see e.g. [KL]; assertion (6) can be obtained by a simple computation.

Theorem 6.1

(1) *If $A \subseteq [0,1]$ does not contain a perfect subset, then A is a U-set.*

(2) *Every Bernstein set is a U-set.*

(3) *There are two U-sets such that their union is the whole interval* $[0,1]$.

(4) *Every set of cardinality smaller that $\mathfrak{c}$ is a U-set.*

(5) *There exists a U-set which is neither meager nor has measure zero (and so is not σ-porous).*

(6) *Shifts of U-sets are again U-sets.*

What will happen in the case of a nice U-set, say with the property of Baire or being Lebesgue measurable? Using some elementary facts about trigonometric series one can prove the folklore result (for a proof see e.g. [Ba2, KL])

Theorem 6.2 *If a U-set is Lebesgue measurable, then it has measure zero.*

In 1916 D. E. Menchoff [Men] distinguished U-sets and Lebesgue measure zero sets by proving

Theorem 6.3 *There is a perfect set of Lebesgue measure zero which is not a U-set.*

The case of Baire property waited several years for the answer . In 1986 G. Debs and J. Saint-Raymond [DSR], using methods of descriptive set theory, proved

Theorem 6.4 *Every U-set which has the property of Baire is meager.*

The existence of a perfect U-set has been shown by N. K. Bary [Ba1]. Independently, A. Rajchman [Raj] proved

Theorem 6.5 *Every H_σ-set is a U-set.*

In 1952 I. I. Piatetskiĭ-Shapiro [PS] proved that the opposite is not true; actually, he proved the following (see [Ba2]):

Theorem 6.6 *There is a closed U-set which cannot be covered by a sequence of closed porous sets and hence is not an H_σ-set.*

By Theorem 6.1 (3), the union of two U-sets need not be a U-set. However, in some important special cases, it is. N. K. Bary [Ba1] proved that *the union of countably many closed U-sets is also a U-set.*

N. N. Kholshchevnikova [Kh1] remarked that actually

Theorem 6.7 *The union of less than* add($\mathcal{K}$) *closed U-sets is a U-set.*

By Theorem 6.7 the union of two F_σ sets of uniqueness is also a U-set. A partial extension has been proved by N. N. Kholshchevnikova [Kh1]:

The union of two disjoint G_δ U-sets is a U-set.
If G, H are U-sets and G is both F_σ and G_δ, then $G \cup H$ is a U-set.

Thus, a U-set which is simultaneously F_σ and G_δ, is $\mathcal{U}$-permitted. These results were generalized by C. Carlet and G. Debs [CD] as

Theorem 6.8 *Let A_n, $n \in \omega$, be U-sets that are closed relative to the union $A = \bigcup_{n=0}^{\infty} A_n$. Then A is also a U-set.*

Let us remark that every countable set of finite Cantor-Bendixson rank is both F_σ and G_δ. In particular,

Corollary 6.9 *By adding a finite set to a U-set, one again obtains a U-set.*

7 Thin sets related to the convergence and the absolute convergence of trigonometric series

J. Marcinkiewicz [Ma1], in honour of V. V. Niemytzkiĭ, introduced the notion of an N-set (investigated earlier by P. Fatou [Fat] and A. Rajchman [Raj]): a set $A \subseteq [0,1]$ is an ***N-set*** if there is a trigonometric series (1.1) absolutely converging on A with $\sum_{n=0}^{\infty}(|a_n| + |b_n|) = \infty$ (i.e. not converging absolutely everywhere). The family of all N-sets will be denoted by $\mathcal{N}$. In 1941 R. Salem proved the first three parts of the following theorem, the parts (1) and (2) in [Sa1] and the part (3) in [Sa2]; later, J. Arbault [Arb] proved the last one.

Theorem 7.1

(1) *Whenever the set of absolute convergence of the trigonometric series (1.1) is non-empty, then it is a shift of the set of absolute convergence of the series $\sum_{n=1}^{\infty} \rho_n \sin 2\pi n x$, where $\rho_n = \sqrt{a_n^2 + b_n^2}$.*

(2) *A set $A \subseteq [0,1]$ is an N-set if and only if there are non-negative reals ρ_n, $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and the series $\sum_{n=1}^{\infty} \rho_n \sin \pi n x$ absolutely converges for $x \in A$.*

(3) *By adding a point, consequently a finite set, to an N-set, one again obtains an N-set.*

(4) *A set* $A \subseteq [0,1]$ *is an N-set if and only if there are reals* $\rho_n \geq 0, k_n \geq 1$, $n = 1,2,\ldots$, *such that* $\sum_{n=1}^{\infty} \rho_n = \infty$ *and the series* $\sum_{n=1}^{\infty} \rho_n \sin \pi k_n x$ *absolutely converges for* $x \in A$.

Let us recall that a set of reals A is called a **$\mathbb{Z}$-basis** if every real x can be written in the form

$$x = \sum_{i=1}^{n} k_i a_i,$$

for some $a_i \in A$ and suitable integers k_i, $i = 1,\ldots,n$. A well known result of H. Steinhaus says that if a set A contains a Borel subset which either has positive measure or is non-meager, then the set $A - A = \{x - y : x, y \in A\}$ contains an interval and consequently A is a $\mathbb{Z}$-basis. From this point of view, the following result[3] of V. V. Niemytzkiĭ [Nie] is a common extension of theorems 1.3 and 1.4.

Theorem 7.2 *If the series* $\sum_{n=1}^{\infty} b_n \sin 2\pi n x$ *absolutely converges on a* $\mathbb{Z}$-*basis, then* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Combining this result with Theorem 7.1 we obtain

Corollary 7.3

(1) *A* $\mathbb{Z}$-*basis is not an N-set.*

(2) *Every N-set is meager and has Lebesgue measure zero.*

(3) *Shifts and expansions of N-sets are N-sets again.*

Then, J. Arbault [Arb] and independently P. Erdös[4] proved

Arbault-Erdös Theorem 7.4 *By adding a countable set to an N-set, one again obtains an N-set.*

J. Arbault [Arb] remarks that when constructing an N-set, one usually chooses the coefficients as 0 or 1. Therefore, he defined: a set $A \subseteq [0,1]$ is called an **N_0-set** if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that

$$\sum_{k=0}^{\infty} \sin \pi n_k x \tag{7.1}$$

[3] Perhaps it is due to the above mentioned result on $\mathbb{Z}$-bases why J. Arbault [Arb] attributes this theorem to H. Steinhaus.

[4] J. Arbault in [Arb, pp. 271–2] writes that "M. R. Salem m'a signalé que ce théorème a été démontré par M. P. Erdös, mais non publié." Let us remark that the notes of J. Arbault about the Erdös' proof are sufficient for reconstructing it.

converges absolutely for every $x \in A$. As before, $\mathcal{N}_0$ will denote the family of all N_0-sets. Evidently, every N_0-set is an N-set. He also introduced the notion of a set "admettend suite de limite nulle" that was named later A-set in his honour: a set A is called an ***A-set*** if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{\sin n_k \pi x\}_{k=0}^{\infty}$ converges to 0 for every $x \in A$. It is easy to see that

$$\mathcal{N}_0 \subseteq \mathcal{A} \subseteq \mathcal{H}_\sigma.$$

In [Arb] he gave the following answers to fundamental questions about N_0-sets and A-sets:

Theorem 7.5

(1) *Every countable set of reals is an N_0-set.*

(2) *By adding a point to an N_0-set or an A-set, one obtains an N_0-set or an A-set, respectively.*

(3) *Shifts and expansions of N_0-sets and A-sets are N_0-sets and A-sets, respectively.*

Moreover, he gives an important example

Theorem 7.6 *If a_n are positive reals, $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$, then the N-set*

$$\{x \in [0,1] : \sum_{n=1}^{\infty} a_n |\sin 2^n \pi x| < \infty\}$$

is not an A-set. Consequently, there exists an N-set that is not an N_0-set.

By the Piatetskiĭ-Shapiro Theorem 3.1, every A-set is σ-porous. In 1985 S. V. Konyagin (unpublished?) showed that (for a proof see [Zaj])

Theorem 7.7 *The N-set*

$$\{x \in [0,1] : \sum_{n=1}^{\infty} \frac{1}{n} |\sin n! \pi x| \leq 1\}$$

is not σ-porous.

It is easy to see that Niemytzkiĭ's result can be extended to A-sets: a $\mathbb{Z}$-basis is not an A-set. H. Steinhaus proved that the Cantor set is a $\mathbb{Z}$-basis. J. Arbault showed that the Cantor set cannot be covered by countable many N-sets. On the other hand, the Cantor set is an H-set and therefore porous. Thus

$$\mathcal{P} \not\subseteq \mathcal{N}_\sigma, \quad \mathcal{P} \not\subseteq \mathcal{A}, \quad \mathcal{H} \not\subseteq \mathcal{A}.$$

In connection with the Cantor-Lebesgue Theorem 1.2, R. Salem [Sa3] introduced the notion of an R-set (R in honour of A. Rajchman according to his result quoted below): a set A is called an ***R-set*** if there is a trigonometric series (1.1) converging on A with coefficients not converging to zero; $\mathcal{R}$ is the family of all R-sets.

A. Rajchman [Raj] has shown (using other notions) that

every R-set is an H_σ-set.

In 1990 S. Kahane [Ka2] proved that $\mathcal{R} \subseteq \mathcal{A}$. In 1991 S. V. Konyagin[5] [Kon] proved the converse. Thus

Theorem 7.8 $\mathcal{A} = \mathcal{R}$.

Using this result, N. N. Kholshchevnikova [Kh4] proved

Theorem 7.9 *By adding a countable set to an A-set one obtains again an A-set.*

From now on, we will prefer the name "A-set".

8 Other trigonometric thin sets

A set A is called a ***Dirichlet set*** or shortly a ***D-set*** if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\{\sin n_k \pi x\}_{k=0}^{\infty}$ uniformly converges to zero on A. The family of all Dirichlet sets is denoted by $\mathcal{D}$. A set A is called an ***almost Dirichlet set*** or shortly an ***aD-set*** if every proper subset B of A which is closed in A is a Dirichlet set: the corresponding family is denoted by $a\mathcal{D}$. A set A is called a ***pseudo Dirichlet set***[6] or shortly a ***pD-set*** if there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\{\sin n_k \pi x\}_{k=0}^{\infty}$ quasinormally converges to zero on A. The corresponding family is denoted by $p\mathcal{D}$. By Theorem 2.1 we have

$$p\mathcal{D} \subseteq \mathcal{N}_0.$$

Evidently, every Dirichlet set is an almost Dirichlet set, i.e.

$$\mathcal{D} \subseteq a\mathcal{D}.$$

One can easily see that every Dirichlet set is an H-set. By the definition of an almost Dirichlet set, small neighbourhoods of points are Dirichlet sets and

[5] The title of S. V. Konyagin's paper [Kon] is misleading—he proves actually the opposite inclusion.

[6] This notion has been introduced under the name D-set in [BZ1]. S. Kahane [Ka2] independently introduced the notion of a pseudo Dirichlet set.

so porous by Theorem 3.1. Since porosity at a point is a local property of a set, we obtain that every almost Dirichlet set is porous[7].

Z. Bukovská [BZ1] proved

Theorem 8.1

(1) *A set A is pseudo Dirichlet if and only if it is the union of an increasing sequence of Dirichlet sets.*

(2) $a\mathcal{D} \subseteq p\mathcal{D} \subseteq \mathcal{D}_\sigma$.

(3) *Every almost Dirichlet subset of* $[0,1]$ *which is a subgroup of* $\mathbb{R}/\mathbb{Z}$ *is finite.*

By Theorem 2.2 and Theorem 8.1 we obtain (the proof of the fifth part is similar to that of Theorem 7.5 (3), see [Arb])

Theorem 8.2

(1) *Every finite set is Dirichlet.*

(2) *Every countable set is pseudo Dirichlet.*

(3) *Adding a finite set to a Dirichlet set one obtains a Dirichlet set.*

(4) *Adding a countable set to a pseudo Dirichlet set one obtains a pseudo Dirichlet set.*

(5) *Shifts and expansions of Dirichlet sets and pseudo Dirichlet sets are Dirichlet sets and pseudo Dirichlet sets, respectively.*

We do not know to whom the following result should be ascribed. For a proof, see e.g. [Kah].

Theorem 8.3 *If* $P_1, \dots, P_n$ *are pairwise disjoint perfect subsets of* $[0,1]$, *then there exists a Dirichlet (even Kronecker) set P such that* $P \cap P_i$ *is (non-empty) perfect for* $i = 1, \dots, n$.

Summarizing the preceding results, we obtain the following chain of inclusions:

$$\mathcal{D} \subseteq a\mathcal{D} \subseteq p\mathcal{D} \subseteq \mathcal{N}_0 \subseteq \mathcal{A} = \mathcal{R} \subseteq \mathcal{H}_\sigma \subseteq \mathcal{U}. \tag{8.1}$$

Let us note that every H_σ-set is σ-porous and so is meager and has measure zero. All inclusions in the chain (8.1) are proper. T. W. Körner [Kor]

[7] Actually, Dirichlet sets are strongly symmetrically porous and hence almost Dirichlet sets are such too.

constructed a perfect non-Dirichlet set, all of whose proper closed subsets are Dirichlet (even Kronecker). By theorems 8.1 (3) and 8.2 (2), every infinite countable subgroup of $\mathbb{R}/\mathbb{Z}$ is a pseudo Dirichlet set which is not an almost Dirichlet set.

J. Arbault [Arb] proved

Theorem 8.4 *The A-set*

$$\{x \in [0,1] : \lim_{n\to\infty} \sin 2^{2^n}\pi x = 0\} \tag{8.2}$$

in not an N-set. Moreover, its subset

$$\{x \in [0,1] : \sum_{n=0}^{\infty} |\sin 2^{2^n}\pi x|^2 < \infty\}$$

is not an N_0-set.

S. Kahane [Ka2] proved

Theorem 8.5 *If the increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ is such that $\lim_{k\to\infty}(n_{k+1} - n_k) = \infty$, then the compact N_0-set*

$$\{x \in [0,1] : \sum_{k=0}^{\infty} |\sin 2^{n_k}\pi x| \leq 1\}$$

is not in $\mathcal{D}_\sigma$ and therefore not a pseudo Dirichlet set. Moreover, if the sequence $\{n_{k+1} - n_k\}_{k=0}^{\infty}$ is strictly increasing, then the A-set

$$\{x \in [0,1] : \sum_{k=0}^{\infty} \sin 2^{n_k}\pi x \ \textit{converges}\}$$

is not an N_σ-set.

By theorems 7.6 and 8.5, we have

$$\mathcal{N}_0 \subsetneq \mathcal{N} \not\subseteq \mathcal{A}, \quad \mathcal{A} \not\subseteq \mathcal{N}$$

and by theorems 3.1 and 7.7, we have

$$\mathcal{H}_\sigma \subsetneq \mathcal{P}_\sigma, \quad \mathcal{N} \not\subseteq \mathcal{P}_\sigma.$$

If $L \subseteq \omega$, we denote

$$K_L = \{x \in [0,1] : \text{there are } x_i = 0,1 \text{ such that } x = \textstyle\sum_{i\in L} \frac{x_i}{2^{i+1}}\}.$$

One can easily see that the set K_L is perfect, assuming that L is infinite. The union $K_L \cup K_{\omega\setminus L}$ is a $\mathbb{Z}$-basis. It is easy to see that if the complement of L contains segments of consecutive integers of unbounded length (such a set is called colacunary), then K_L is a Dirichlet set. So, we obtain the result proved by J. Marcinkiewicz [Ma2]:

Marcinkiewicz Theorem 8.6 *One can choose the set L in such a way that both sets K_L and $K_{\omega\setminus L}$ are D-sets.*

Corollary 8.7 *There are two perfect D-sets such that their union is a $\mathbb{Z}$-basis and so neither an N-set nor an A-set.*

To define a new type of thin set, we first recall that a Borel measure μ on $[0, 1]$ is a finite σ-additive measure defined on a σ-algebra $\mathcal{S}$ containing all Borel sets. We assume that all Borel measures are complete (see e.g. [Frl]), in the sense that every set of outer μ-measure zero is in $\mathcal{S}$. A set $A \subseteq [0, 1]$ is universally measurable iff A is measurable for every Borel measure on $[0, 1]$. In particular, every analytic and so every Borel set is universally measurable. A set $A \subseteq [0, 1]$ is said to have universal measure zero if for each non-atomic, non-negative Borel measure μ on $[0, 1]$, $\mu(A) = 0$.

We define the notion of a ***weak Dirichlet set*** in two steps:

i) A universally measurable set $A \subseteq [0, 1]$ is weak Dirichlet if for every positive Borel measure μ on $[0, 1]$, there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_A |e^{2\pi i n_k x} - 1| \, d\mu(x) = 0.$$

ii) Generally, a (non-universally measurable) set is weak Dirichlet if it is contained in some universally measurable weak Dirichlet set.

The family of all weak Dirichlet sets will be denoted by $w\mathcal{D}$. This definition of weak Dirichlet set was introduced by S. Kahane [Ka2]. B. Host, J.-F. Méla and F. Parreau [HMP] introduced this notion only for Borel sets, T. W. Körner [Kor] only for closed sets. Notice that D-sets and pD-sets are just the sets on which some sequence $\{e^{2\pi i n_k x} - 1\}_{k=0}^{\infty}$ converges to 0 uniformly and quasinormally, respectively.

Directly from the definition of a weak Dirichlet set, we obtain (part (1) is mentioned in [BZ1, Ka2], see also [Kor])

Theorem 8.8

(1) *Every A-set is weak Dirichlet.*

(2) *Every universal measure zero set is $w\mathcal{D}$-permitted.*

The following result on weak Dirichlet sets is folklore and the proof uses ideas which were known already to J. Arbault. The proofs of all analogical results for D-sets, N_0-sets and A-sets are almost the same—the only difference is in the choice of a convenient convergence of functions. This is one reason that we give its proof here, the other is that we are not able to give a complete reference (closure under shifts is proved e.g. in [Ka1]).

Theorem 8.9 *Shifts and expansions of weak Dirichlet sets are weak Dirichlet sets.*

PROOF. It is easy to see that 0 can always be added to a weak Dirichlet set. Hence, closure under adding a point will be a consequence of closure under translations. We use the equality $|e^{2\pi i nx} - 1| = 2|\sin \pi n x|$. Let $A \subseteq [0,1]$ be a weak Dirichlet set, $x \neq 0$ a real number, and $\lim_{k\to\infty} \int_A |\sin n_k \pi y|\, d\mu(y) = 0$. We can easily find an increasing subsequence $\{n_{k_j}\}_{j=0}^{\infty}$ of the sequence $\{n_k\}_{k=0}^{\infty}$ such that $\|(n_{k_{j+1}} - n_{k_j})x\| < 2^{-j}$ and $\|(n_{k_{j+1}} - n_{k_j})x^{-1}\| < 2^{-j}$. Now for each $y \in A$,

$$\begin{aligned} |\sin(n_{k_{j+1}} - n_{k_j})\pi(x+y)| &\leq \\ &\leq |\sin(n_{k_{j+1}} - n_{k_j})\pi x| + |\sin n_{k_{j+1}}\pi y| + |\sin n_{k_j}\pi y| \\ &\leq 2^{-j}\pi + |\sin n_{k_{j+1}}\pi y| + |\sin n_{k_j}\pi y|. \end{aligned}$$

Integrating over $y \in A$ and taking the limit as $j \to \infty$, we get that the set $x + A$ is a weak Dirichlet set.

Let m_j be the nearest integer to $(n_{k_{j+1}} - n_{k_j})x^{-1}$, i.e. $\|(n_{k_{j+1}} - n_{k_j})x^{-1}\| = |m_j - (n_{k_{j+1}} - n_{k_j})x^{-1}|$. We can choose n_{k_j} so that the sequence $\{m_j\}_{j=0}^{\infty}$ is increasing. Now for each $y \in A$, $|xy| \leq |x|$ and

$$\begin{aligned} |\sin m_j \pi x y| &\leq |\sin(n_{k_{j+1}} - n_{k_j})\pi y| + |\sin \|(n_{k_{j+1}} - n_{k_j})x^{-1}\|\pi x y| \\ &\leq |\sin n_{k_{j+1}}\pi y| + |\sin n_{k_j}\pi y| + 2^{-j}\pi|x|. \end{aligned}$$

Hence by integration and taking limit in these inequalities we get that xA is a weak Dirichlet set.

□

B. Host, J.-F. Méla and F. Parreau [HMP] proved the following

Theorem 8.10 *A Borel set A is an N-set if and only if there exists an F_σ weak Dirichlet set B containing A as a subset.*

Corollary 8.11 $\mathcal{N} \cap \mathcal{F}_\sigma = w\mathcal{D} \cap \mathcal{F}_\sigma$.

Let us remark that from this corollary we can obtain many known results on N-sets, e.g. Salem theorems 7.1 (1), (3), and the Arbault-Erdös Theorem 7.4.

In [Ka2], the following result of G. Debs is presented:

Theorem 8.12 *If A is an analytic ($= \Sigma^1_1$) weak Dirichlet set, then the group generated by the set A is also a weak Dirichlet set.*

By Theorem 8.12 and Corollary 8.7, we immediately get:

Corollary 8.13

(1) *Every analytic weak Dirichlet set is meager and has Lebesgue measure zero.*

(2) *There are two perfect Dirichlet sets whose union is not a weak Dirichlet set. Consequently, $\mathcal{D}_\sigma \not\subseteq w\mathcal{D}$.*

In Section 9 we show that the word "analytic" cannot be omitted in the above corollary.

9 Borel bases

By a simple computation, you can see that every D-set is contained in a closed D-set. Similarly, every pD-, N_0- and N-set is contained in an F_σ set of the same family. An A-set is always contained in an $F_{\sigma\delta}$ A-set. Thus, the family $\mathcal{D}$ has closed basis, the families $p\mathcal{D}$, $\mathcal{N}_0$ and $\mathcal{N}$ have F_σ bases, and the family $\mathcal{A}$ has an $F_{\sigma\delta}$ basis.

Can these computations be improved? In other words, can we find bases of these families consisting of simpler Borel sets? At least in a certain sense, the answer is "No".

The case of closed sets cannot be improved by open ones, since open sets are not small (e.g. they have positive measure). The case of F_σ and $F_{\sigma\delta}$ sets cannot be improved by G_δ sets: the set $[0,1]\cap\mathbb{Q}$ is an F_σ pseudo Dirichlet set, and every G_δ set containing it is not meager and therefore neither an N-set nor an A-set.

By Theorem 8.10, any A-set which is not an N-set is not contained in any F_σ weak Dirichlet set. In particular, the A-set (8.2) is not contained in any F_σ A-set.

We show that it is consistent with ZFC that $w\mathcal{D}$ does not have a Borel basis. Let $\mathfrak{M}$ be a transitive model of ZFC and let c be a Cohen real over $\mathfrak{M}$ (for the details see e.g. [Jech]). It is well known that the set $A = [0,1]\cap\mathfrak{M}$ of the reals of the ground model has universal measure zero and is not meager in the generic extension $\mathfrak{M}[c]$. So, the set A is weak Dirichlet and, by Corollary 8.13, cannot be contained in an analytic weak Dirichlet set.

10 A systematic approach

The definitions of D-, pD-, N_0-, N-, R- and A-sets have a common structure: there exists some sequence of functions which converges on the set in a certain sense. In his thesis, P. Eliaš [Eli] investigated the possibilities of defining thin sets related to absolute convergence rather systematically. He considered nine types of conditions on a sequence $\{f_k\}_{k=0}^{\infty}$ of functions defined on a set A:

(P) $\{f_k\}_{k=0}^{\infty}$ converges pointwise to 0 on A,

(QN) $\{f_k\}_{k=0}^{\infty}$ quasinormally converges to 0 on A,

(U) $\{f_k\}_{k=0}^{\infty}$ uniformly converges to 0 on A,

(PS) $\sum_{k=0}^{\infty} f_k(x)$ converges pointwise on A,

(QNS) the sequence of partial sums of the series $\sum_{k=0}^{\infty} f_k(x)$ quasinormally converges on A,

(US) the sequence of partial sums of the series $\sum_{k=0}^{\infty} f_k(x)$ uniformly converges on A,

(PNS) $\sum_{k=0}^{\infty} f_k(x)$ pseudonormally converges on A, i.e. there is a sequence of positive reals $\{\epsilon_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} \epsilon_k < +\infty$ and $(\forall x \in A)(\exists k_0)$ $(\forall k \geq k_0)\, |f_k(x)| \leq \epsilon_k$,

(NS) $\sum_{k=0}^{\infty} f_k(x)$ normally converges on A, i.e. there is a sequence of positive reals $\{\epsilon_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} \epsilon_k < +\infty$ and $(\forall x \in A)(\forall k)\, |f_k(x)| \leq \epsilon_k$,

(BS) $\sum_{k=0}^{\infty} f_k(x)$ is bounded on A.

Four types of sequences of functions $\{f_k\}_{k=0}^{\infty}$ are considered:

(S_1) $f_k(x) = |\sin n_k \pi x|$ for some increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers,

(S_2) there exists a sequence $\{a_k\}_{k=0}^{\infty}$ of non-negative real numbers such that $\sum_{k=0}^{\infty} a_k = +\infty$ and $f_k(x) = a_k|\sin k\pi x|$,

(S_3) there exists a sequence $\{a_k\}_{k=0}^{\infty}$ of non-negative real numbers such that $\limsup_{k\to\infty} a_k > 0$ and $f_k(x) = a_k|\sin k\pi x|$,

(S_4) there exists a sequence $\{a_k\}_{k=0}^{\infty}$ of non-negative real numbers such that $\limsup_{k\to\infty} a_k = +\infty$ and $f_k(x) = a_k|\sin k\pi x|$.

For example, a set A is a pD-set if and only if there exists a sequence of type (S_1) satisfying the condition (QN). In this way, we obtain 36 definitions. However, it turns out that many of them are equivalent (e.g., (S_1) is in our context equivalent to (S_3)), and almost all of them were at least implicitly known. P. Eliaš [Eli] explicitly defined two new classes of thin sets which were implicitly considered already by S. Kahane [Ka2]: a set A is a ***B_0-set*** if there exists a sequence of type (S_1) satisfying the condition (BS). A set A is a ***B-set*** if there exists a sequence of type (S_2) satisfying the condition (BS). Evidently,

$$\mathcal{B}_0 \subseteq \mathcal{B}, \quad \mathcal{B}_0 \subseteq \mathcal{N}_0, \quad \mathcal{B} \subseteq \mathcal{N}.$$

Modifying some classical reasoning, P. Eliaš [Eli] proved

Theorem 10.1

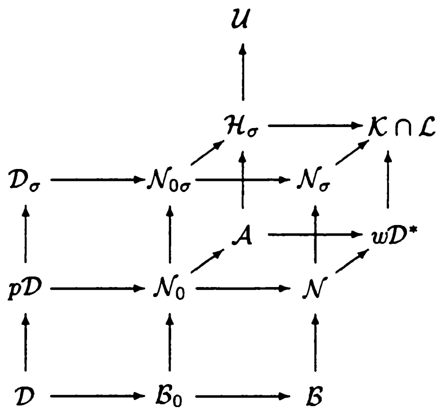
(1) *By adding a point to a B_0-set or a B-set, one obtains a B_0-set or a B-set, respectively.*

(2) *Shifts and expansions of B_0-sets and B-sets are B_0-sets and B-sets, respectively.*

The main result of [Eli] can be expressed by the following table, which gives the family of thin sets (or the family of all subsets of $[0,1]$, denoted by "all") corresponding to each combination of sequence type (S_i) and convergence condition:

	(P)	(QN)	(U)	(PS)	(QNS)	(US)	(PNS)	(NS)	(BS)
(S_1)	$\mathcal{A}$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{N}_0$	$p\mathcal{D}$	$\mathcal{D}$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$
(S_2)	all	all	all	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{B}_0$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}_0$
(S_3)	$\mathcal{A}$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{N}_0$	$p\mathcal{D}$	$\mathcal{D}$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$
(S_4)	$p\mathcal{D}$	$p\mathcal{D}$	$\mathcal{D}$	$p\mathcal{D}$	$p\mathcal{D}$	$\mathcal{D}$	$p\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$

The relationships between these families is given by the following diagram, where the arrow '$\rightarrow$' means the inclusion '$\subseteq$' and $w\mathcal{D}^*$ denotes the family of all sets contained in a Σ^1_1 weak Dirichlet set.



Additional families:
$\mathcal{B}_{0\sigma} = \mathcal{N}_{0\sigma}$
$\mathcal{B}_\sigma = \mathcal{N}_\sigma$
$\mathcal{D} \subseteq a\mathcal{D} \subseteq p\mathcal{D}$
$a\mathcal{D} \subseteq \mathcal{P}$
$\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{P}$
$\mathcal{H}_\sigma \subseteq \mathcal{P}_\sigma \subseteq \mathcal{K} \cap \mathcal{L}$

Note that the restriction to $w\mathcal{D}^*$ in the above picture is necessary since every Luzin set which is non-meager while having strong measure zero is a weak Dirichlet set. Recall that an uncountable set X is a Luzin set if every meager subset of X is countable. Note also that, assuming the continuum hypothesis, there is a Luzin set $X \subseteq [0,1]$ such that $X - X = [0,1]$. In particular this means that in Theorem 8.12, the restriction to Σ^1_1 sets cannot be dropped.

In fact, all the inclusions in this diagram are proper, and no other inclusions between the families included in the diagram hold true. This is a consequence of these six inequalities: $\mathcal{B}_0 \not\subseteq \mathcal{D}_\sigma$ (Theorem 8.5), $p\mathcal{D} \not\subseteq \mathcal{B}$ (by Theorem 8.2 (2) the set $\mathbb{Q} \cap [0,1]$ is pseudo Dirichlet and is not a B-set, since the closure of a B-set is a B-set again), $\mathcal{D}_\sigma \not\subseteq w\mathcal{D}$ (Corollary 8.13 (2)), $\mathcal{B} \not\subseteq \mathcal{H}_\sigma$ (Theorem 7.7, since $\mathcal{H}_\sigma \subseteq \mathcal{P}_\sigma$), $\mathcal{A} \not\subseteq \mathcal{N}_\sigma$ (Theorem 8.5), and $\mathcal{U} \not\subseteq \mathcal{H}_\sigma$ (Theorem 6.6). One can also easily see that all the inclusions between the additional families are proper.

To be sure that we did not forget any possible inclusion between these families, we will use this auxiliary notion: a family $\mathcal{X}$ in the diagram is said to be OK if for every family $\mathcal{Y}$ in the diagram the inclusion $\mathcal{X} \subseteq \mathcal{Y}$ holds just in the case that there is a path $\mathcal{X} \to \cdots \to \mathcal{Y}$ in the diagram.

We start with some simple facts, proving that the families $\mathcal{D}$, $\mathcal{B}_0$, $p\mathcal{D}$, $\mathcal{D}_\sigma$, $\mathcal{B}$, $\mathcal{A}$, $\mathcal{U}$ are OK:

$\mathcal{D}$ is OK.

$\mathcal{B}_0 \not\subseteq \mathcal{D}_\sigma$ implies $\mathcal{B}_0$ is OK.

$p\mathcal{D} \not\subseteq \mathcal{B}$ implies $p\mathcal{D}$ is OK.

$\mathcal{D}_\sigma \not\subseteq w\mathcal{D}$ implies $\mathcal{D}_\sigma$ is OK.

$\mathcal{B} \not\subseteq \mathcal{H}_\sigma$ implies $\mathcal{B}$ is OK.

$\mathcal{A} \not\subseteq \mathcal{N}_\sigma$ implies $\mathcal{A}$ is OK.

$\mathcal{U} \not\subseteq \mathcal{H}_\sigma$ implies $\mathcal{U}$ is OK.

Now notice that for each family $\mathcal{X}$ in the diagram, either by one of the previous simple facts, $\mathcal{X}$ is OK, or there are at least two families $\mathcal{Y}, \mathcal{Z}$ so that the arrows $\mathcal{Y} \to \mathcal{X}$, $\mathcal{Z} \to \mathcal{X}$ are in the diagram. Moreover, whenever both $\mathcal{Y}, \mathcal{Z}$ are OK, then (and this can be easily verified directly in the diagram) also $\mathcal{X}$ is OK. Hence, using this property of the diagram and induction, we can successively prove that every family in the diagram is OK.

11 Ten questions

Let $\mathcal{F}$ be a family of thin subsets of the unit interval $[0,1]$. We ask the following questions:

$Q1$ Is $\mathcal{F}$ an ideal?

$Q2$ Does $\mathcal{F}$ have a Borel basis?

$Q3$ Is $\mathcal{F}$ a subfamily of $\mathcal{K}$?

$Q4$ Is $\mathcal{F}$ a subfamily of $\mathcal{L}$?

$Q5$ Is every set in $\mathcal{F}$ σ-porous?

$Q6$ Does $\mathcal{F}$ contain a perfect set?

$Q7$ Is every countable subset of $[0,1]$ in $\mathcal{F}$?

$Q8$ Is for every $A \in \mathcal{F}$ and every $x \in [0,1]$ the union $A \cup \{x\}$ also in $\mathcal{F}$?

$Q9$ Is for every $A \in \mathcal{F}$ and every real x the shift $x + A$ also in $\mathcal{F}$?

$Q10$ Is for every $A \in \mathcal{F}$ and every real x the expansion xA also in $\mathcal{F}$?

Since any family $\mathcal{F}$ of thin sets satisfies conditions a) and b) of the definition of an ideal, question $Q1$ is equivalent to the question of whether $\mathcal{F}$ satisfies condition c). Therefore, question $Q1$ is often referred to as the union problem.

We raise another set of questions: what are the sizes of the cardinal characteristics of considered families of thin sets; i.e., what are the cardinals $\mathrm{non}(\mathcal{F})$, $\mathrm{add}(\mathcal{F})$, $\mathrm{cov}(\mathcal{F})$ and $\mathrm{cof}(\mathcal{F})$ for the investigated families $\mathcal{F}$ of thin sets of harmonic analysis?

The table on page 481 gives complete answers (except one) to questions $Q1$–10 about the nine families of thin sets. However, the computation of cardinal characteristics in the following sections is far from complete.

12 Replacing countable

The past twenty years of investigations in set theory have showed that very often the word "countable" can be replaced by "less than a small cardinal characterizing the structure of $\mathcal{P}(\omega)$". In Section 4, we introduced the small cardinals $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{h}$, $\mathfrak{b}$ and $\mathfrak{d}$. It turns out that they play an important role in the study of trigonometric thin sets. Now, we present the main recent results of this kind.

Our story begins in 1985, when N. N. Kholshchevnikova [Kh2] improved Arbault's Theorem 7.5 (1) by showing that

every set of cardinality smaller than $\mathfrak{m}$ is an N_0-set.

<table>
<tr><th>$\mathcal{Q}$</th><th>$\mathcal{D}$</th><th>$p\mathcal{D}$</th><th>$\mathcal{N}_0$</th><th>$\mathcal{A}$</th><th>$\mathcal{N}$</th><th>$w\mathcal{D}$</th><th>$\mathcal{B}_0$</th><th>$\mathcal{B}$</th><th>$\mathcal{U}$</th></tr>
<tr><td rowspan="2">1</td><td>No</td><td>No</td><td>No</td><td>No</td><td>No</td><td>No</td><td>No</td><td>No</td><td>No</td></tr>
<tr><td colspan="8">8.13 (2) (see also 8.7)</td><td>6.1 (3)</td></tr>
<tr><td rowspan="2">2</td><td>closed</td><td>F_σ</td><td>F_σ</td><td>$F_{\sigma\delta}$</td><td>F_σ</td><td>$\frac{1}{2}$No[8]</td><td>closed</td><td>closed</td><td>No</td></tr>
<tr><td colspan="8">Section 9</td><td>6.1 (2)</td></tr>
<tr><td rowspan="2">3</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>No[9]</td></tr>
<tr><td colspan="8">8.13 (1) (see also 1.4 and 3.1)</td><td>6.1 (5)</td></tr>
<tr><td rowspan="2">4</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>No[9]</td></tr>
<tr><td colspan="8">8.13 (1) (see also 1.3 and 3.1)</td><td>6.1 (5)</td></tr>
<tr><td rowspan="2">5</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>No</td><td>No</td><td>Yes</td><td>No</td><td>No</td></tr>
<tr><td colspan="4">3.1</td><td colspan="2">7.7</td><td>3.1</td><td>7.7</td><td>6.1 (5)</td></tr>
<tr><td rowspan="2">6</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td></tr>
<tr><td colspan="8">8.3</td><td>6.5</td></tr>
<tr><td rowspan="2">7</td><td>No</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>No</td><td>No</td><td>Yes</td></tr>
<tr><td>$\mathbb{Q}$</td><td colspan="5">8.2 (2)</td><td>$\mathbb{Q}$</td><td>$\mathbb{Q}$</td><td>1.1</td></tr>
<tr><td rowspan="2">8</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td></tr>
<tr><td>8.2 (3)</td><td>8.2 (4)</td><td colspan="2">7.5 (2)</td><td>7.1 (3)</td><td>8.8 (2)</td><td colspan="2">10.1 (1)</td><td>6.9</td></tr>
<tr><td rowspan="2">9</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td></tr>
<tr><td colspan="2">8.2 (5)</td><td colspan="2">7.5 (3)</td><td>7.3 (3)</td><td>8.9</td><td colspan="2">10.1 (2)</td><td>6.1 (6)</td></tr>
<tr><td rowspan="2">10</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>Yes</td><td>No[10]</td></tr>
<tr><td colspan="2">8.2 (5)</td><td colspan="2">7.5 (3)</td><td>7.3 (3)</td><td>8.9</td><td colspan="2">10.1 (2)</td><td>[Ba2]</td></tr>
</table>

[8] "$\frac{1}{2}$No" means that the negative answer is consistent with ZFC.

[9] Compare with Theorem 6.2 and Theorem 6.4, respectively.

[10] If $E \subseteq [0,1]$ is a U-set and for each $y \in E$, $xy \in [0,1]$, then the expansion xE is a U-set (see [MZ] or [Ba2]). Generally this is not true: Let E_1, E_2 be Bernstein sets covering the interval $[0,1]$. Then the set $E = (1/2E_1) \cup (1/2 + 1/2E_2)$ is a U-set but $2E = [0,1)$.

In 1990 Z. Bukovská [BZ1] replaced N_0-set by pseudo Dirichlet set and the cardinal $\mathfrak{m}$ by the "greater" cardinal $\mathfrak{p}$ in this result. Actually, she proved a stronger result:

Theorem 12.1 *Let $\{B_s : s \in S\}$ be a family of Dirichlet sets. If $|S| < \mathfrak{p}$ and for every finite $T \subseteq S$ the union $\bigcup_{s\in T} B_s$ is a Dirichlet set, then the union $\bigcup_{s\in S} B_s$ is a pseudo Dirichlet set.*

As a corollary, we obtain (see [BB]) that by adding a set of cardinality smaller than $\mathfrak{p}$ to a pD-set one obtains a pD-set, i.e.

$$\mathrm{non}(\mathrm{Prm}(p\mathcal{D})) \geq \mathfrak{p}.$$

T. Bartoszyński and M. Scheepers [BS] improved these results by showing that

$$\mathrm{non}(\mathrm{Prm}(p\mathcal{D})) \geq \mathfrak{h}, \quad \mathrm{non}(\mathrm{Prm}(\mathcal{N}_0)) \geq \mathfrak{h}. \tag{12.1}$$

L. Bukovský, I. Recław and M. Repický [BRR] considered topological spaces (and sets of reals), not distinguishing between pointwise and quasinormal convergence of real valued functions. Let us recall the main notion of this paper: a set $X \subseteq [0,1]$ is called a ***wQN-set*** if for every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real-valued functions defined on X and converging to 0 on X, there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the subsequence $\{f_{n_k}(x)\}_{k=0}^{\infty}$ quasinormally converges to 0 on X. Let us recall that the inclusions

$$w\mathcal{QN} \cap \mathcal{A} \subseteq p\mathcal{D} \subseteq \mathcal{N}_0$$

were the motivation for introducing the notion of a wQN-set in [BRR]. We can prove more.

Theorem 12.2

(1) *If $E \in \mathcal{N}_0$ and X is a wQN-set with $|X| < \mathfrak{s}$, then $E \cup X \in \mathcal{N}_0$.*

(2) *If $E \in p\mathcal{D}$ and X is a wQN-set with $|X| < \mathfrak{s}$, then $E \cup X \in p\mathcal{D}$.*

PROOF. (1) Let $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of integers such that the series (7.1) absolutely converges for $x \in E$.

By Corollary 4.1 to Booth's Lemma, there exists a subsequence $\{m_k\}_{k=0}^{\infty}$ of the sequence $\{n_k\}_{k=0}^{\infty}$ such that both sequences

$$\{\sin m_k \pi x\}_{k=0}^{\infty}, \quad \{\cos m_k \pi x\}_{k=0}^{\infty}$$

converge on the set X. Without loss of generality we can assume that the sequence $i_k = m_{k+1} - m_k$, $k = 0, 1, \ldots$, is strictly increasing. Taking limits in the equality

$$\sin i_k \pi x = \sin m_{k+1} \pi x \, \cos m_k \pi x - \sin m_k \pi x \, \cos m_{k+1} \pi x, \tag{12.2}$$

we obtain $\lim_{k\to\infty} \sin i_k \pi x = 0$ for $x \in X$. Since X is a wQN-set, by Theorem 2.1 there is a subsequence $\{j_k\}_{k=0}^{\infty}$ of $\{i_k\}_{k=0}^{\infty}$ such that the series

$$\sum_{k=0}^{\infty} \sin j_k \pi x \tag{12.3}$$

absolutely converges for $x \in X$. Using the inequality

$$|\sin i_k \pi x| \leq |\sin m_{k+1} \pi x| + |\sin m_k \pi x|,$$

one can easily see that the series (12.3) converges absolutely also on the set E.

(2) This proof can be deduced from the previous one by replacing each instance of "the absolute convergence of a series" by "the quasinormal convergence of a sequence".

□

Every set of reals of cardinality smaller than $\mathfrak{b}$ is a wQN-set (see [BRR, p. 35]). Therefore

Corollary 12.3

(1) $\mathrm{non}(\mathrm{Prm}(p\mathcal{D})) \geq \min\{\mathfrak{s}, \mathfrak{b}\}$,

(2) $\mathrm{non}(\mathrm{Prm}(\mathcal{N}_0)) \geq \min\{\mathfrak{s}, \mathfrak{b}\}$.

Since there is a model of ZFC in which $\mathfrak{h} < \min\{\mathfrak{s}, \mathfrak{b}\}$, [She], this strengthens (12.1). The hypothesis, "countable" in the Arbault-Erdös Theorem 7.4 and in Theorem 7.9 was replaced with a small cardinal by Z. Bukovská and L. Bukovský [BB] and N. N. Kholshchevnikova [Kh2, Kh5]:

$$\mathrm{non}(\mathrm{Prm}(\mathcal{N})) \geq \mathfrak{p}, \quad \mathrm{non}(\mathrm{Prm}(\mathcal{A})) \geq \mathfrak{m}.$$

T. Bartoszyński and M. Scheepers [BS] improved these inequalities as follows:

Theorem 12.4

(1) $\mathrm{non}(\mathrm{Prm}(\mathcal{N})) \geq \mathfrak{t}$,

(2) $\mathrm{non}(\mathrm{Prm}(\mathcal{A})) \geq \mathfrak{s}$.

F. Hausdorff [Hau] constructed a universal measure zero set of cardinality $\aleph_1$. A. W. Miller [Mil] proved that every set of cardinality smaller than $\mathrm{cov}(\mathcal{K})$ has strong measure zero. Since every strong measure zero set has universal measure zero (see [Lav]), by Theorem 8.8 (2) we obtain

Theorem 12.5

(1) *There is an uncountable $w\mathcal{D}$-permitted set.*

(2) $\mathrm{non}(\mathrm{Prm}(w\mathcal{D})) \geq \mathrm{cov}(\mathcal{K})$.

For the considered families of thin sets, we cannot say whether there exists a permitted set of power $\mathfrak{c}$. A partial answer—a consistency result—will be given in the next section. J. Arbault [Arb] presented a theorem saying that there exists a perfect $\mathcal{N}$-permitted set. However, N. K. Bary [Ba2] has found a gap in his proof.

Now we present some upper estimates for covering numbers of the families $\mathcal{A}$ and $\mathcal{D}$. We need one more small uncountable cardinal.

A family $\mathcal{F}$ of subsets of ω is said to be a ***refining family*** if for every $A \subseteq \omega$ there exists a $B \in \mathcal{F}$ such that $B \subseteq^* A$ or $B \subseteq^* \omega \setminus A$. Thus, $\mathfrak{r}$ is the least size of a refining family. A related small cardinal was defined by P. Vojtáš [Vo1]: $\mathfrak{r}_\sigma$ is the least size of a family $\mathcal{F} \subseteq [\omega]^\omega$ such that for every sequence A_n, $n \in \omega$ of subsets of ω there exists a $B \in \mathcal{F}$ such that for every $n \in \omega$, either $B \subseteq^* A_n$ or $B \subseteq^* \omega \setminus A_n$. We say that $\mathcal{F}$ is a ***σ-refining family.*** It is known [Vo1, Vau] that $\mathfrak{r} \leq \mathfrak{r}_\sigma$ and that ZFC + $(\mathfrak{d} < \mathfrak{c})$ + $(\mathfrak{r}_\sigma < \mathfrak{c})$ is consistent.

Theorem 12.6

(1) $\mathrm{cov}(\mathcal{A}) \leq \mathfrak{r}_\sigma$,

(2) $\mathrm{cov}(\mathcal{D}) \leq \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.

PROOF. Let us recall that for an infinite set $L \subseteq \omega$, $L(n)$ denotes the n^{th} member of L. Let $\mathcal{F}$ be a σ-refining family of cardinality $\mathfrak{r}_\sigma$. We can assume that for each $L \in \mathcal{F}$ the sequence $\{L(n+1) - L(n)\}_{n=0}^{\infty}$ is strictly increasing. For such L, we consider the A-set

$$V_L = \{y \in [0,1] : \lim_{n\to\infty} \sin(L(n+1) - L(n))\pi x = 0\}.$$

(1) For $x \in [0,1]$ and $q \in [-1,1] \cap \mathbb{Q}$ we define

$$L_{x,q} = \{k \in \omega : \sin k\pi x \leq q\}, \quad K_x = \{k \in \omega : \cos k\pi x \geq 0\}.$$

Let $x \in [0,1]$ be fixed. Then there is an $L \in \mathcal{F}$ such that for all $q \in [-1,1] \cap \mathbb{Q}$, either $L \subseteq^* L_{x,q}$ or $L \subseteq^* \omega \setminus L_{x,q}$ and either $L \subseteq^* K_x$ or $L \subseteq^* \omega \setminus K_x$. Similarly, as in the proof of Booth's Lemma 4.1, one can show that $\{\sin k\pi x\}_{k\in L}$ converges. Moreover, the sequence $\{\cos k\pi x\}_{k\in L}$ does not change sign and therefore, also converges. Using equality (12.2) with $m_k = L(k)$, we obtain that x belongs to the set V_L.

Hence $[0,1] = \bigcup_{L\in\mathcal{F}} V_L$ is the union of $\mathfrak{r}_\sigma$-many A-sets.

(2) Let $\mathcal{H} \subseteq {}^\omega\omega$ be a dominating family of size $\mathfrak{d}$. For $f \in \mathcal{H}$, $L \in \mathcal{F}$, and $k \in \omega$ we consider the D-set

$$Z_{L,f,k} = \{y \in V_L : (\forall n \geq k)\,|\sin(L(f(n)+1) - L(f(n)))\pi y| \leq 1/(n+1)\}.$$

For each $x \in [0,1]$, there is an $L \in \mathcal{F}$ such that $x \in V_L$. We set

$$g(n) = \min\{k : (\forall m \geq k)\,|\sin(L(m+1) - L(m))\pi x| \leq 1/(n+1)\}.$$

Since $\mathcal{H}$ is a dominating family, there exists an $f \in \mathcal{H}$ such that $g \leq^* f$; i.e., there exists a $k \in \omega$ such that $g(n) \leq f(n)$ for every $n \geq k$. Then $x \in Z_{L,f,k}$.

Now the proof is finished since we have

$$[0,1] = \bigcup_{L\in\mathcal{F}} \bigcup_{f\in\mathcal{H}} \bigcup_{k\in\omega} Z_{L,f,k}$$

and $|\mathcal{F} \times \mathcal{H} \times \omega| = \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.

□

13 γ-sets are permitted

If X is a subset of $[0,1]$ (or more generally, a topological space), we consider the set $C(X)$ of continuous real-valued functions defined on X with the topology inherited from the product space ${}^X\mathbb{R}$. There exists a basis of this topology consisting of the sets

$$\{g \in C(X) : |g(x_i) - f(x_i)| < \varepsilon_i,\ \text{for } i = 0,\ldots,n\},$$

where $f \in C(X)$, $x_i \in X$, $\varepsilon_i > 0$, $i = 0,\ldots,n$, and $n \in \omega$. Moreover, a sequence $\{f_n\}_{n=0}^\infty$ of functions from $C(X)$ converges to a function $f \in C(X)$ in this topology if and only if it does so pointwise on X.

F. Gerlits and Z. Nagy [GN] introduced the notion of a γ-set. A set $X \subseteq [0,1]$ is called a ***γ-set*** if $C(X)$ is a Fréchet space; i.e., if for every subset A of $C(X)$ and every f in the closure of A, there exists a sequence of elements of A converging (pointwise) to f. A family $\mathcal{V}$ of subsets of $[0,1]$ is called an ***ω-cover*** of a set X if for every finite set $X_0 \subseteq X$ there is a set $V \in \mathcal{V}$ such that $X_0 \subseteq V$. For the proof of the following characterization of γ-sets see [GN].

Theorem 13.1 *A set X is a γ-set if and only if for every open ω-cover $\mathcal{V}$ of X there is a sequence $\{V_k\}_{k\in\omega}$ of sets from $\mathcal{V}$ such that $X \subseteq \bigcup_{m=0}^\infty \bigcap_{k=m}^\infty V_k$.*

According to [GN] and [GM], we know that

$$\text{non}(\gamma\text{-sets}) = \mathfrak{p}.$$

F. Galvin and A. W. Miller [GM] proved

Theorem 13.2 *If $\mathfrak{p} = \mathfrak{c}$, then there exists a γ-set of cardinality $\mathfrak{c}$.*

The proof of the next result is based on the ideas of J. Arbault [Arb] (see also [BB]).

Theorem 13.3 *γ-sets are $\mathcal{N}$-permitted.*

PROOF. Let X be an infinite γ-set, and let $\{y_k\}_{k=0}^{\infty}$ be a sequence of distinct elements of X. Let E be the set of absolute convergence of a series $\sum_{n=1}^{\infty} \rho_n |\sin n\pi x|$ with $\sum_{n=1}^{\infty} \rho_n = \infty$. We prove that $E \cup X$ is an N-set.

Set $S_n = \sum_{k=1}^{n} \rho_k$. By applying the integral criterion for convergence and divergence of series to $f(x) = 1/x$ and $f(x) = 1/(x^{1+1/p})$ we have

$$\sum_{n=1}^{\infty} \frac{\rho_n}{S_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\rho_n}{S_n^{1+\frac{1}{p}}} < \infty, \quad \text{for each } p > 0.$$

We can easily find a monotone unbounded sequence of integers $\{p_n\}_{n=1}^{\infty}$ such that (compare with [Ba2, Zy1])

$$\sum_{n=1}^{\infty} \frac{\rho_n}{S_n^{1+\frac{1}{p_n}}} < \infty.$$

Let us define $\rho'_n = \rho_n / S_n$, $\varepsilon_n = 1/S_n^{1/p_n}$ and $g(n) = \min\{m : \sum_{k=n}^{m} \rho'_k \geq 1\}$. By Theorem 2.2, for any reals $x_1, \dots, x_{p_n} \in [0,1]$, there is an integer $k_n \leq S_n = (1/\varepsilon_n)^{p_n}$ such that $|\sin k_n n\pi x_i| < 2\pi\varepsilon_n$, for $i = 1, 2, \dots, p_n$. For integer k, let Ω_k be the set of all finite sequences of integers $\lambda(k), \lambda(k+1), \dots, \lambda(g(k))$ such that $\lambda(n) \leq S_n$ for $n = k, k+1, \dots, g(k)$. For $k \in \omega$ and $\lambda \in \Omega_k$, let

$$U_{\lambda,k} = \{x \in [0,1] : (\forall n \in [k, g(k)] \cap \omega)\, |\sin \lambda(n) n\pi x| < 2\pi\varepsilon_n\}$$

and

$$V_{\lambda,k} = U_{\lambda,k} \setminus \{y_k\}.$$

Clearly, the family $\mathcal{V} = \{V_{\lambda,k} : k \in \omega \;\&\; \lambda \in \Omega_k\}$ is an open ω-cover. Hence, there is a sequence $\{(\lambda_k, n_k)\}_{k=0}^{\infty}$ such that $X \subseteq \bigcup_{m=0}^{\infty} \bigcap_{k=m}^{\infty} V_{\lambda_k, n_k}$. As $y_n \in X$, the equality $n = n_k$ can hold true for at most finitely many $k \in \omega$. Hence, without loss of generality we can assume that $n_{k+1} > g(n_k)$ for all $k \in \omega$. We prove that the series

$$\sum_{k=0}^{\infty} \sum_{n=n_k}^{g(n_k)} \rho'_n |\sin \lambda_k(n) n\pi x| \tag{13.1}$$

converges on $E \cup X$, and so $E \cup X$ is an N-set.

For $x \in E$,

$$\rho'_n |\sin \lambda_k(n) n\pi x| \leq \rho'_n S_n |\sin n\pi x| = \rho_n |\sin n\pi x|$$

and so the series (13.1) converges.

For each $x \in X$ there is an m such that $(\forall k \geq m)\, x \in V_{\lambda_k, n_k}$. Hence,

$$\sum_{k=m}^{\infty} \sum_{n=n_k}^{g(n_k)} \rho'_n |\sin \lambda_k(n) n\pi x| \leq \sum_{k=m}^{\infty} \sum_{n=n_k}^{g(n_k)} \rho'_n 2\pi\varepsilon_n = 2\pi \sum_{k=m}^{\infty} \sum_{n=n_k}^{g(n_k)} \frac{\rho_n}{S_n^{1+\frac{1}{\rho_n}}} < \infty.$$

□

We shall need the following fact [BRR]:

every γ-set is both a wQN-set and a pseudo Dirichlet set.

Now we prove

Theorem 13.4 *Every γ-set is permitted for the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{A}$ and $w\mathcal{D}$.*

PROOF. Assume that X is a γ-set and that E is a pD-set. There is a sequence $\{||n_k x||\}_{k=0}^{\infty}$ quasinormally converging to 0 on E. We can assume that $n_j - n_i = n_k - n_l$ if and only if $j = k$ and $i = l$. Thus, the set of all differences $\{n_j - n_i : (\exists k)\, k \leq i < j \leq 2k\}$ can be simply ordered as an increasing sequence $\{m_k\}_{k=0}^{\infty}$. By Theorem 2.2, the 0-function belongs to the closure of the set $\{||m_k x|| : k \in \omega\} \subseteq C(X)$. Since X is a γ-set, there exists a subsequence of this sequence converging pointwise to 0. Without loss of generality we can assume that $||m_k x|| \to 0$ pointwise on X. Since X is a wQN-set, there exists a subsequence of $\{||m_k x||\}_{k=0}^{\infty}$ converging quasinormally to 0 on X. Again, we can assume that $||m_k x|| \to 0$ quasinormally on X. Every m_k is of the form $n_{j_k} - n_{i_k}$, $i_k < j_k \leq 2i_k$, and so each i can repeat only finitely many times in the sequence $\{i_k\}_{k=0}^{\infty}$. Hence without loss of generality, we can assume that the sequences $\{i_k\}_{k=0}^{\infty}$ and $\{j_k\}_{k=0}^{\infty}$ are both increasing. Therefore, since $||m_k x|| \leq ||n_{j_k} x|| + ||n_{i_k} x||$, $||m_k x|| \to 0$ quasinormally also on the union $E \cup X$.

In the cases of E being an N_0- or A-set, the proofs proceed in the same way.

Every γ-set has strong measure zero and consequently universal measure zero. Hence by Theorem 8.8 (2) it is $w\mathcal{D}$-permitted.

□

Corollary 13.5 *One cannot prove that there is no $p\mathcal{D}$-, $\mathcal{N}_0$-, $\mathcal{N}$-, $\mathcal{A}$- or $w\mathcal{D}$-permitted set of cardinality $\mathfrak{c}$ (provided that ZFC is consistent).*

PROOF. Assuming that ZFC is consistent, there is a model of ZFC in which $\mathfrak{p} = \mathfrak{c}$, (see e.g. [Jech]). By Theorem 13.2, in this model there are γ-sets of cardinality $\mathfrak{c}$ which are, according to theorems 13.3 and 13.4, $p\mathcal{D}$-, $\mathcal{N}_0$-, $\mathcal{N}$-, $\mathcal{A}$- and $w\mathcal{D}$-permitted sets.

□

Unfortunately, wQN-sets are perfectly meager ([BRR, p. 31]), and so we did not obtain an example of a perfect permitted set. Moreover, every γ-set has strong measure zero [GN]. Thus, in Laver's model [Lav], every γ-set is countable. Hence, γ-sets are not the tool for finding big permitted sets in ZFC alone.

14 Rademacher orthogonal system

Rademacher orthogonal system is the sequence $\mathfrak{R} = \{r_n\}_{n=0}^{\infty}$ of functions

$$r_n(x) = \operatorname{sgn}(\sin 2^n \pi x), \quad \text{for } x \in [0,1].$$

For information about the properties of the Rademacher system, we recommend e.g. [Ale, Ba2, Zyl].

For a real x, we denote

$$S_x = \{i \in \omega : r_i(x) = -1\}.$$

S is a mapping from $[0,1)$ onto $\{A \in \mathcal{P}(\omega \setminus \{0\}) : \omega \setminus A \text{ is infinite}\}$ and it is one-to-one on the set of all non-dyadic reals.

For a real $x \in [0,1]$, we denote by $x(i)$ the i^{th} digit in the dyadic expansion of x; i.e.,

$$x = \sum_{i=1}^{\infty} x(i) 2^{-i}.$$

If $x \neq 1$ is a dyadic real, for clarity we assume that $x = \sum_{i=1}^{k} x(i)2^{-i}$, with $x(k) = 1$ and $x(i) = 0$ for $i \geq k$. Then for $n \geq 1$, we obtain

$$r_n(x) = \begin{cases} (-1)^{x(n)}, & \text{for } n = 1, \ldots, k-1, \\ 0, & \text{for } n \geq k. \end{cases}$$

If x is a non-dyadic real then $r_n(x) = (-1)^{x(n)}$ for all $n \geq 1$. Therefore, for a dyadic real x, S_x is finite; and for a non-dyadic real x,

$$S_x = \{i \in \omega : x(i) = 1\}.$$

Since $|r_n(x)| = 1$ for every non-dyadic $x \in [0,1]$, the possibilities of defining thin sets for the Rademacher system are limited. We introduce the following one: a set $E \subseteq [0,1]$ is an $\boldsymbol{A}^{\mathfrak{R}}$***-set*** if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\{r_{n_k}(x)\}_{k=0}^{\infty}$ converges for every $x \in E$.

For an infinite subset L of ω, we denote

$$X_L = \{x \in [0,1] : L \subseteq^* S_x \text{ or } L \subseteq^* \omega \setminus S_x\}.$$

Therefore, $x \in X_L$ if and only if the sequence $\{x(i)\}_{i \in L}$ is eventually constant (i.e. constant for every i greater than some k) or, equivalently, if and only if S_x does not split the set L. In particular X_L contains all dyadic reals. This immediately implies that

$$X_L \subseteq X_K \textit{ if and only if } K \subseteq^* L$$

The sequence $\{r_i(x)\}_{i \in L}$ converges if and only if the sequence $\{x(i)\}_{i \in L}$ is eventually constant, and therefore we have

Lemma 14.1

(1) *A is an $A^{\mathfrak{R}}$-set if and only if there exists an $L \in [\omega]^{\omega}$ such that $A \subseteq X_L$.*

(2) *Every set X_L is F_σ.*

Let $x \in X_L$ be a non-dyadic real. Then either for all but finitely many $n \in L$, $x(n) = 1$ and $\{2^{n-1}x\} = \sum_{i=0}^{\infty} x(n+i)2^{-i-1} > 1/2$ (and so $\{2^{n-1}x - 1/2\} < 1/2$), or for all but finitely many $n \in L$, $x(n) = 0$ and $\{2^{n-1}x\} = \sum_{i=1}^{\infty} x(n+i)2^{-i-1} < 1/2$. It follows that X_L is an H_σ-set

Thus, we can summarize:

Theorem 14.2

(1) *Every $A^{\mathfrak{R}}$-set is an H_σ-set and therefore meager, negligible and σ-porous.*

(2) *The family $\{X_L : L \in [\omega]^{\omega}\}$ is an F_σ basis of $\mathcal{A}^{\mathfrak{R}}$.*

(3) *There are perfect $A^{\mathfrak{R}}$-sets.*

(4) *A set A is $\mathcal{A}^{\mathfrak{R}}$-permitted if and only if for every infinite $L \subseteq \omega$ there exists an infinite $K \subseteq L$ such that $A \subseteq X_K$.*

Next we show

Theorem 14.3

(1) $\mathrm{non}(\mathcal{A}^{\mathfrak{R}}) = \mathfrak{s}$,

(2) $\mathrm{cov}(\mathcal{A}^{\mathfrak{R}}) = \mathfrak{r}$.

PROOF. (1) It follows immediately from the definitions that every set A with $|A| < \mathfrak{s}$ is an $\mathrm{A}^{\mathfrak{R}}$-set.

Conversely, let $\mathcal{F}$, $|\mathcal{F}| = \mathfrak{s}$, be a splitting family; i.e., every infinite subset of ω is split by some set from $\mathcal{F}$. One can easily check that the set

$$\left\{\sum_{i\in K\setminus\{0\}} 2^{-i} : K \in \mathcal{F}\right\}$$

is not a subset of any X_L and therefore not an $\mathrm{A}^{\mathfrak{R}}$-set.

(2) Let $\mathcal{F}$ be a refining family of the cardinality $\mathfrak{r}$. Let $x \in [0,1]$. Then there exists a set $L \in \mathcal{F}$ such that $L \subseteq^* S_x$ or $L \subseteq^* \omega \setminus S_x$. In both cases $x \in X_L$. Therefore

$$[0,1] = \bigcup_{L\in\mathcal{F}} X_L,$$

and so $\mathrm{cov}(\mathcal{A}^{\mathfrak{R}}) \leq \mathfrak{r}$.

Conversely, assume that $\mathcal{C} \subseteq \mathcal{A}^{\mathfrak{R}}$ covers the interval $[0,1]$. We can assume that $\mathcal{C} = \{X_L : L \in \mathcal{F}\}$ where $\mathcal{F} \subseteq [\omega]^\omega$. We show that $\mathcal{F}$ is a refining family. Let $K \in [\omega]^\omega$ be such that $\omega \setminus K$ is infinite. We set $x = \sum_{i\in K\setminus\{0\}} 2^{-i}$. Then there is a set $L \in \mathcal{F}$ such that $x \in X_L$, and so $x(i)$ is either 1 for all but finitely many $i \in L$ or 0 for all but finitely many $i \in L$. It follows that $L \subseteq^* K$ or $L \subseteq^* \omega \setminus K$. Hence $\mathfrak{r} \leq \mathrm{cov}(\mathcal{A}^{\mathfrak{R}})$.

□

We introduce a new small cardinal:

$$\mathfrak{r}' = \min\{|\mathfrak{K}| : (\forall L \in [\omega]^\omega)(\exists \mathcal{F} \in \mathfrak{K})(\mathcal{F} \text{ is dense in } [\omega]^\omega, \subseteq^* \text{ and } (\forall K \in \mathcal{F})(K \subseteq^* L \text{ or } K \subseteq^* \omega \setminus L))\}.$$

If you choose one element from every $\mathcal{F} \in \mathfrak{K}$, you obtain a refining family. Thus $\mathfrak{r} \leq \mathfrak{r}'$.

Theorem 14.4

(1) $\mathrm{non}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})) = \mathfrak{s}$,

(2) $\mathrm{cov}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})) = \mathfrak{r}'$,

(3) $\mathfrak{h} \leq \mathrm{add}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})) \leq \mathfrak{r}'$.

PROOF. (1) Let $X \subseteq [0,1]$, $|X| < \mathfrak{s}$. Let $A \in \mathcal{A}^{\mathfrak{R}}$. By Theorem 14.2 (2) there exists an infinite $K \subseteq \omega$ such that $A \subseteq X_K$. Since $\{S_x \cap K : x \in X\}$ cannot be a splitting family on K, there exists an $L \in [\omega]^\omega$ with $L \subseteq K$ such that for every $x \in X$ either $L \subseteq^* S_x \cap K$ or $L \subseteq^* K \setminus S_x$. In both cases, $x \in X_L$;

i.e., $X \subseteq X_L$. Since $X_K \subseteq X_L$, we obtain $X \cup A \subseteq X_L \in \mathcal{A}^{\mathfrak{N}}$. The reverse inequality follows from Theorem 14.3 (1).

(2) By Theorem 14.2 (4), for $X \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$ the set

$$\mathcal{F}(X) = \{L \in [\omega]^\omega : X \subseteq X_L\}$$

is an open dense subset of $[\omega]^\omega, \subseteq^*$. If $[0,1] = \bigcup_{\xi \in \kappa} X_\xi$, $X_\xi \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, then the family $\{\mathcal{F}(X_\xi) : \xi \in \kappa\}$ satisfies the condition of the definition of $\mathfrak{r}'$, i.e. $\kappa \geq \mathfrak{r}'$.

Conversely, if $\mathcal{F} \subseteq [\omega]^\omega$ is dense, then the set

$$X(\mathcal{F}) = \{x \in [0,1] : (\forall L \in \mathcal{F})\, L \subseteq^* S_x \text{ or } L \subseteq^* \omega \setminus S_x\}$$

is $\mathcal{A}^{\mathfrak{N}}$-permitted. If $\mathfrak{K}$ is a family from the definition of $\mathfrak{r}'$, then one can easily see that

$$\bigcup_{\mathcal{F} \in \mathfrak{K}} X(\mathcal{F}) = [0,1],$$

i.e. $\mathrm{cov}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})) \leq \mathfrak{r}'$.

(3) Let $\mathcal{X} \subseteq \mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $|\mathcal{X}| < \mathfrak{h}$. By the definition of $\mathfrak{h}$, the set $\mathcal{F} = \bigcap_{X \in \mathcal{X}} \mathcal{F}(X)$ is a dense subset of $[\omega]^\omega$. Since clearly $\bigcup \mathcal{X} \subseteq X(\mathcal{F})$, the set $\bigcup \mathcal{X}$ is $\mathcal{A}^{\mathfrak{N}}$-permitted, and therefore $\mathrm{add}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})) \geq \mathfrak{h}$.

The second inequality follows from part (2).

□

We show that there are perfect $\mathcal{A}^{\mathfrak{N}}$-permitted sets. We start with an auxiliary result.

Lemma 14.5 *Let $\mathcal{F} \subseteq [\omega]^\omega$ and let $x_L \in X_{\omega \setminus L}$ for each $L \in \mathcal{F}$. If $\mathcal{F}$ is an almost disjoint family (i.e. $K \cap L$ is finite for any different $K, L \in \mathcal{F}$), then the set $\{x_L : L \in \mathcal{F}\}$ is $\mathcal{A}^{\mathfrak{N}}$-permitted.*

PROOF. Let $A \in \mathcal{A}^{\mathfrak{N}}$. Then there exists a $K \in [\omega]^\omega$ such that $A \subseteq X_K$. We have two possibilities.

If for every $L \in \mathcal{F}$ the intersection $L \cap K$ is finite, then we set $N = K$. Otherwise, there exists an $M \in \mathcal{F}$ such that the intersection $K \cap M$ is infinite. In this case, we take an infinite set $N \subseteq K \cap M$ such that $\{x_M(i)\}_{i \in N}$ is constant on N, i.e. $x_M \in X_N$.

In the former case for any $L \in \mathcal{F}$ and in the latter case for any $L \in \mathcal{F}$ except M, the intersection $N \cap L$ is finite and therefore $N \subseteq^* \omega \setminus L$. Thus, $x_L \in X_{\omega \setminus L} \subseteq X_N$ for every $L \in \mathcal{F}$.

□

Theorem 14.6 *There exists a perfect $\mathcal{A}^{\mathfrak{N}}$-permitted set.*

PROOF. Let us fix an enumeration $\{s_i : i \in \omega\}$ of the set ${}^{<\omega}2$ of all finite sequences of 0's and 1's. For $\alpha \in {}^{\omega}2$, we denote $C_\alpha = \{i \in \omega : s_i \subseteq \alpha\}$. For any $\beta \neq \alpha$, the intersection $C_\alpha \cap C_\beta$ is finite. Thus, $\{C_\alpha : \alpha \in {}^{\omega}2\}$ is an almost disjoint family.

Let $g : {}^{\omega}2 \to [0,1]$ be defined by $g(\alpha) = \sum_{i \in C_\alpha \setminus \{0\}} 2^{-i}$. Then $g(\alpha) \in X_{\omega \setminus C_\alpha}$ for every $\alpha \in {}^{\omega}2$, and the set $\{g(\alpha) : \alpha \in {}^{\omega}2\}$, as a one-to-one continuous image of a compact space (the set ${}^{\omega}2$ is endowed with the product topology), is perfect and by Lemma 14.5 also $\mathcal{A}^{\mathfrak{R}}$-permitted.

□

For $\mathrm{A}^{\mathfrak{R}}$-sets, we know more. However, the proof uses some deep methods of logic (absoluteness). For the notion of a Mathias real, see e.g. [Mat].

Theorem 14.7 *Each perfect set P contains a perfect $A^{\mathfrak{R}}$-subset.*

PROOF. Let $\mathfrak{M}$ be a transitive model of ZFC containing P and let $\mathbf{m}$ be a Mathias real over $\mathfrak{M}$. Then in $\mathfrak{M}[\mathbf{m}]$ there is a $K \in [\omega]^\omega$ such that

$$(\forall x \in P \cap \mathfrak{M})(K \subseteq^* S_x \text{ or } K \subseteq^* \omega \setminus S_x).$$

Since $P \cap \mathfrak{M}$ is uncountable in $\mathfrak{M}[\mathbf{m}]$, the Borel set

$$\{x \in P : K \subseteq^* S_x \text{ or } K \subseteq^* \omega \setminus S_x\}$$

is uncountable in $\mathfrak{M}[\mathbf{m}]$. Therefore it contains a perfect subset. Thus we have shown that in $\mathfrak{M}[\mathbf{m}]$ the following formula holds:

$$(\exists K \in [\omega]^\omega)(\exists P' \text{ perfect})(\forall x \in P')[P' \subseteq P \;\&\; (K \subseteq^* S_x \text{ or } K \subseteq^* \omega \setminus S_x)].$$

As this formula is Σ^1_2, by the Shoenfield Absoluteness Lemma (see e.g. [Jech]), it also holds true in $\mathfrak{M}$.

□

15 Consistency of $\mathfrak{r}' < \mathfrak{c}$

About the cardinal characteristic $\mathfrak{r}'$, at the moment, we only know that $\mathfrak{r} \leq \mathfrak{r}'$. Hence, the equality $\mathrm{cov}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})) = \aleph_1$ is not provable. We show that the equality $\mathrm{cov}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})) = \mathfrak{c}$ also cannot be proved in ZFC. To do this, we describe a generic model of ZFC in which $\mathfrak{r}' = \aleph_1 < \mathfrak{c}$. For terminology, see e.g., [Jech].

Let $\mathfrak{M}$ be a transitive model of ZFC + $\aleph_1 < \mathfrak{c}$. Given an ultrafilter $\mathcal{V} \subseteq \mathcal{P}(\omega)$, we consider the forcing notion

$$P(\mathcal{V}) = \{(a, L) : a \subseteq \omega \text{ finite}, \; L \in \mathcal{V}\}$$

with the ordering

$$(s, L) \leq (t, K) \text{ iff } s \supseteq t \ \& \ (s \setminus t) \cup L \subseteq K.$$

This forcing notion is C.C.C., so cardinals are not collapsed when forcing with it. If G is an $\mathfrak{M}$-generic filter on $P(\mathcal{V})$, then the infinite set

$$N_G = \bigcup \{s : (\exists L)\,(s, L) \in G\}$$

is a pseudo-intersection of $\mathcal{V}$. Consequently,

$$(\forall L \in \mathcal{P}(\omega) \cap \mathfrak{M})\, N_G \subseteq^* L \text{ or } N_G \subseteq^* \omega \setminus L.$$

For any infinite $L \subseteq \omega$, $L \in \mathfrak{M}$, if $f_L \in \mathfrak{M}$ is a one-to-one function from ω onto L (e.g. $f_L(n) = L(n)$), then the set $N_{G,L} = f_L(N_G) \subseteq L$ is in $\mathfrak{M}[G]$ and is such that

$$(15.1) \qquad (\forall K \in \mathcal{P}(\omega) \cap \mathfrak{M})\, N_{G,L} \subseteq^* K \text{ or } N_{G,L} \subseteq^* \omega \setminus K.$$

Now we construct a sequence of models by finite support iteration:

$$(15.2) \qquad \langle \mathfrak{M}_\xi : \xi < \omega_1 \rangle$$

such that

(i) $\mathfrak{M}_0 = \mathfrak{M}$,

(ii) $\mathfrak{M}_{\xi+1} = \mathfrak{M}_\xi[G_\xi]$, where G_ξ is an $\mathfrak{M}_\xi$-generic filter over $P(\mathcal{V}_\xi)$ where $\mathcal{V}_\xi \in \mathfrak{M}_\xi$ is an ultrafilter on ω, and

(iii) for ξ limit, $\mathfrak{M}_\xi$ is the finite support iteration limit of the sequence $\langle \mathfrak{M}_\eta : \eta < \xi \rangle$.

Let $\mathfrak{N}$ be the model of ZFC which is the limit of the chain (15.2). Then $\mathfrak{N}$ and $\mathfrak{M}$ have the same cardinals and $\aleph_1 < \mathfrak{c}$ in $\mathfrak{N}$.

Now set

$$\mathfrak{K}_\eta = \{N_{G_\xi, L} : L \in [\omega]^\omega \cap \mathfrak{M}_\xi \ \& \ \eta \leq \xi\}.$$

Since for each $K \in \mathfrak{N}$, $K \in [\omega]^\omega$, there exists a $\xi < \omega_1$ such that $K \in \mathfrak{M}_\xi$, all the sets $\mathfrak{K}_\eta$, $\eta < \omega_1$, are dense subsets of $([\omega]^\omega)^{\mathfrak{N}}$, and using (15.1) we can easily verify that the family $\{\mathfrak{K}_\eta : \eta < \omega_1\}$ witnesses the equality $\mathfrak{r}' = \aleph_1$ in $\mathfrak{N}$.

16 Walsh orthogonal system

We denote the support of a natural number n by

$$S_n = \text{the unique finite set } L \text{ such that } n = \sum_{i\in L} 2^i.$$

Thus, e.g. $S_0 = \emptyset$, $S_{2^n} = \{n\}$.

The **Walsh orthogonal system** is the sequence $\mathfrak{W} = \{w_n\}_{n=0}^{\infty}$ of functions defined by

$$w_n(x) = \prod_{k\in S_n} r_{k+1}(x), \quad \text{for a non-dyadic } x \in [0,1],$$

and satisfying the equality

$$w_n(x) = \frac{1}{2} \lim_{h\to 0+} (w_n(x-h) + w_n(x+h))$$

for all $x \in [0,1]$, where $w_n(0-h) = w_n(1-h)$ and $w_n(1+h) = w_n(0+h)$. It is known (see e.g. [Ale]) that the system $\mathfrak{W}$ is a complete orthonormal system in $L^2([0,1])$. Clearly $r_{n+1} = w_{2^n}$ and $r_0(x) = w_0(x)$ for $x \in (0,1)$.

As for the Rademacher system, we say that a set $E \subseteq [0,1]$ is an ***A***$^{\mathfrak{W}}$***-set*** if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\{w_{n_k}(x)\}_{k=0}^{\infty}$ converges for every $x \in E$. As before, the family of all A$^{\mathfrak{W}}$-sets is denoted by $\mathcal{A}^{\mathfrak{W}}$.

The values of Rademacher functions for a non-dyadic real x are determined by the set S_x. Let us introduce similar sets for Walsh functions:

$$T_x = \{n \in \omega : w_n(x) = -1\}.$$

Thus, for any non-dyadic real x we have

$$w_n(x) = \begin{cases} +1, & \text{if } n \notin T_x, \\ -1, & \text{if } n \in T_x. \end{cases}$$

Instead of the set X_L, for an infinite $L \subseteq \omega$ we introduce the set

$$Y_L = \{x \in [0,1] : L \subseteq^* T_x \vee L \subseteq^* \omega \setminus T_x\}.$$

We begin with showing that the set Y_L is σ-porous. Let

$$Y_{L,n}^+ = \{x \in [0,1] : L \setminus n \subseteq T_x\},$$
$$Y_{L,n}^- = \{x \in [0,1] : L \setminus n \subseteq \omega \setminus T_x\}.$$

Clearly $Y_L = \bigcup_{n\in\omega} Y_{L,n}^+ \cup Y_{L,n}^-$. Let $n_k = L(k)$ and let $m_k = \max S_{n_k}$. Without loss of generality, we can assume that $\{m_k\}_{k=0}^{\infty}$ is strictly increasing.

Given $n \in \omega$ and any $k > n$ (note that $m_k \geq k > n$) divide the interval $[0, 1]$ into 2^{m_k} equal intervals. Then for each three neighbouring intervals, one is disjoint from $Y^+_{L,n}$ and one is disjoint from $Y^-_{L,n}$. Therefore, all the sets $Y^+_{L,n}$, $Y^-_{L,n}$ are porous, and consequently Y_L is σ-porous.

Lemma 16.1 *The family*

$$\mathcal{W} = \{L \in [\omega]^\omega : \{w_n(x)\}_{n \in L} \textit{ converges for every dyadic } x \in [0, 1]\}$$

is an open dense subset of $[\omega]^\omega, \subseteq^*$.

PROOF. For $x \in [0, 1]$, the set $\mathcal{W}_x$ of all sets $L \in [\omega]^\omega$ such that $\{w_n(x)\}_{n \in L}$ converges is an open dense subset of $[\omega]^\omega, \subseteq^*$. So the set $\mathcal{W}$, being a countable intersection of open dense sets $\mathcal{W}_x$ for x dyadic, is open dense (see e.g. [Vau]). □

Now one can easily prove

Theorem 16.2

(1) $\mathcal{A}^{\mathfrak{R}} \subseteq \mathcal{A}^{\mathfrak{W}}$.

(2) $K \subseteq^* L$ *implies* $Y_L \subseteq Y_K$.

(3) $A \in \mathcal{A}^{\mathfrak{W}}$ *if and only if there is an* $L \in \mathcal{W}$ *which is not split by any set* T_x *for* $x \in A$, *i.e. if* $A \subseteq Y_L$.

(4) *Every* Y_L *is a* σ*-porous* F_σ *set and so is meager and negligible.*

(5) *The family* $\{Y_L : L \in \mathcal{W}\}$ *is an* F_σ *basis for* $\mathcal{A}^{\mathfrak{W}}$.

(6) $\mathcal{A}^{\mathfrak{W}} \subseteq \mathcal{P}_\sigma$.

Note that (3) cannot be reversed: take an infinite set $M \subseteq \{2^n : n \in \omega\}$. Then for any sets L, K such that $M \subseteq L, K \subseteq M \cup \{M(k) + M(k+1) + M(k+2) : k \in \omega\}$, we have $Y_L = Y_K = Y_M$ (recall that $M(k)$ is the k^{th} element of M).

There exists a close relationship between $\mathrm{A}^{\mathfrak{R}}$- and $\mathrm{A}^{\mathfrak{W}}$-sets, expressed by

Theorem 16.3 *There is a Borel mapping* $h : [0, 1] \to [0, 1]$ *which is one-to-one on the set of all non-dyadic reals and such that for any* $x \in [0, 1]$ *and any* $L \in \mathcal{W}$

(16.1) $$x \in Y_L \textit{ if and only if } h(x) \in X_L.$$

Therefore,

$$\mathcal{A}^{\mathfrak{W}} = \{h^{-1}(A) : A \in \mathcal{A}^{\mathfrak{R}}\}.$$

PROOF. Let

$$h(x) = \sum_{n \in T_x} 2^{-n}, \quad \text{for } x \in [0,1].$$

If $x, y \in (0,1)$ are two distinct non-dyadic reals, then the sets T_x, T_y are also distinct. As the complement of any of these two sets cannot be a finite set, $h(x) \neq h(y)$. Clearly, a real x is dyadic if and only if $h(x)$ is dyadic, and h restricted to the set of all non-dyadic reals is continuous. Therefore h is Borel measurable. For a non-dyadic real x and $n \geq 1$,

$$r_n(h(x)) = (-1)^{h(x)_n} = \begin{cases} +1, & \text{if } n \notin T_x, \\ -1, & \text{if } n \in T_x, \end{cases}$$

and so $w_n(x) = r_n(h(x))$, for all $n \in \omega$. Consequently, using the fact that X_L and Y_L both contain all dyadic reals for $L \in \mathcal{W}$, we obtain (16.1).

□

Theorem 16.4

(1) $\{h^{-1}(A) : A \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})\} \subseteq \mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$.

(2) *Each perfect set* $P \subseteq [0,1]$ *contains a perfect* $A^{\mathfrak{W}}$*-subset.*

(3) $\mathfrak{s} \leq \mathrm{non}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})) \leq \mathrm{non}(\mathcal{A}^{\mathfrak{W}})$.

(4) $\mathrm{cov}(\mathcal{A}^{\mathfrak{W}}) \leq \dot{\mathrm{cov}}(\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})) \leq \mathfrak{r}'$.

PROOF. (1) If $A \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})$, then $(\forall L \in [\omega]^\omega)(\exists K \in [\omega]^\omega)(A \cup X_L \subseteq X_K)$. We can always choose such a K from the family $\mathcal{W}$. Hence $(\forall L \in [\omega]^\omega)(\exists K \in [\omega]^\omega)(h^{-1}(A) \cup Y_L \subseteq Y_K)$ and so

$$h^{-1}(A) \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{W}}).$$

(2) Let P be a perfect set. By Theorem 16.3, $h(P)$ is an uncountable Borel set, and therefore $h(P)$ contains a perfect subset. By Theorem 14.7, there is a perfect $A^{\mathfrak{R}}$-set $P' \subseteq h(P)$. Hence by (1), $h^{-1}(P')$ is an $A^{\mathfrak{W}}$-set, and since it is uncountable Borel, it contains a perfect subset.

(3) If $|X| < \mathfrak{s}$ then by Theorem 14.4 (1), $h(X) \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})$, and by (1), $X \in \mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$.

(4) If $\mathcal{X} \subseteq \mathrm{Prm}(\mathcal{A}^{\mathfrak{R}})$ is a covering of the interval $[0,1]$, then by (1), $h^{-1}(\mathcal{X}) \subseteq \mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$ is also a covering. Hence the inequality is a consequence of Theorem 14.4 (2).

□

17 Sets of uniqueness for Rademacher and Walsh systems

In Section 6, we defined sets of uniqueness for trigonometric series. Similarly, we define: a set $A \subseteq [0,1]$ is a ***$U^{\mathfrak{R}}$-set*** (a ***$U^{\mathfrak{W}}$-set***) if the only series $\sum_{n=0}^{\infty} a_n r_n(x)$ ($\sum_{n=0}^{\infty} a_n w_n(x)$) converging to zero on the set $[0,1] \setminus A$ is the series with $a_n = 0$ for every $n = 0, 1, \ldots$.

S. B. Stechkin and P. L. Ul'yanov [SU] proved the first part and, more recently, A. V. Bakhsheciyan [Bak] proved the second part of the next theorem.

Theorem 17.1

(1) *Whenever $A \subseteq [0,1]$ and $\mu(A) < 1/2$, A is a $U^{\mathfrak{R}}$-set.*

(2) *Every $U^{\mathfrak{R}}$-set $A \subseteq [0,1]$ of Lebesgue measure < 1 is contained in a $U^{\mathfrak{R}}$-set with countable complement.*

Similar results for category were obtained by J. E. Coury [Cou] (the first part) and by N. N. Kholshchevnikova [Kh5] (the second one):

Theorem 17.2

(1) *Every meager set is a $U^{\mathfrak{R}}$-set.*

(2) *Every set $A \subseteq [0,1]$ which is either meager or $|A| < \mathfrak{c}$ is contained in a $U^{\mathfrak{R}}$-set with countable complement.*

Moreover, N. N. Kholshchevnikova [Kh5] proved

Theorem 17.3

(1) *The set $[0,1] \setminus \{2^{-n} : n \geq 1\}$ is a $U^{\mathfrak{R}}$-set and the set $[0,1] \setminus \{2^{-n} : n \geq 2\}$ is not a $U^{\mathfrak{R}}$-set.*

(2) *A set $A \subseteq [0,1]$ containing all dyadic reals is a $U^{\mathfrak{R}}$-set if and only if there are non-dyadic reals $x_n, y_n \in [0,1] \setminus A$ such that $r_{n+1}(x_n) \neq r_{n+2}(x_n)$ and $r_{n+1}(y_n) = r_{n+2}(y_n)$ for all $n \in \omega$.*

A. A. Shneĭder [Shn2] obtained the first fundamental results about $U^{\mathfrak{W}}$-sets.

Theorem 17.4

(1) *Every countable subset of $[0,1]$ is a $U^{\mathfrak{W}}$-set.*

(2) *Every $U^{\mathfrak{W}}$-set has Lebesgue inner measure zero.*

(3) *There exists a Lebesgue measure zero set that is not a* $U^{\mathfrak{W}}$*-set.*

(4) *There exists a* $U^{\mathfrak{W}}$*-set of cardinality* $\mathfrak{c}$.

(5) *The union of a finite number of closed* $U^{\mathfrak{W}}$*-sets is a* $U^{\mathfrak{W}}$*-set.*

W. R. Wade [Wad] improved part (5) by showing that

the union of a countable number of closed $U^{\mathfrak{W}}$-sets is a $U^{\mathfrak{W}}$-set.

N. N. Kholshchevnikova [Kh4] generalized this result in the style of Theorem 6.8 as

Theorem 17.5 *Let* A_n, $n \in \omega$ *be* $U^{\mathfrak{W}}$*-sets that are closed relative to their union* $A = \bigcup_{n=0}^{\infty} A_n$. *Then* A *is also a* $U^{\mathfrak{W}}$*-set.*

Moreover, N. N. Kholshchevnikova [Kh3] proved an analogous result to Debs and Saint-Raymond's theorem 6.4:

Theorem 17.6 *Every* $U^{\mathfrak{W}}$*-set with the Baire property is meager.*

18 More on non-absolute convergence

The combined results of several authors [Rad, PZ, Kol, Zy2] give the following classical theorem on Rademacher series (μ means the Lebesgue measure; for a proof, see e.g. [Ba2]):

Theorem 18.1 *Let* $\{c_n\}_{n=0}^{\infty}$ *be a sequence of reals. Then the following conditions are equivalent:*

(1) $\sum_{n=0}^{\infty} c_n^2 < \infty$,

(2) $\mu(\{x \in [0,1] : \sum_{n=0}^{\infty} c_n r_n(x) \text{ converges}\}) = 1$,

(3) $\mu(\{x \in [0,1] : \sum_{n=0}^{\infty} c_n r_n(x) \text{ converges}\}) > 0$.

For category, S. Kaczmarz and H. Steinhaus [KS] obtained a similar result:

Theorem 18.2 *The following conditions are equivalent:*

(1) $\sum_{n=0}^{\infty} |c_n| = \infty$,

(2) $\{x \in [0,1] : \sum_{n=0}^{\infty} c_n r_n(x) \text{ converges}\}$ *is meager,*

(3) $\{x \in [0,1] : \sum_{n=0}^{\infty} c_n r_n(x) \text{ converges}\} \neq [0,1]$.

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of Borel measurable functions defined on the interval $[0,1]$ with $|f_n(x)| \leq 1$. For a real $p \geq 1$, we denote by

$$\Sigma^p(\{f_n\}_{n=0}^{\infty})$$

the family of sets $A \subseteq [0,1]$ for which there exists a sequence $\{c_n\}_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty} |c_n|^p = \infty$ such that the series $\sum_{n=0}^{\infty} c_n f_n(x)$ converges for all $x \in A$. Clearly, $\Sigma^{p_2}(\{f_n\}_{n=0}^{\infty}) \subseteq \Sigma^{p_1}(\{f_n\}_{n=0}^{\infty})$, whenever $1 \leq p_1 < p_2$.

Using the result of A. A. Shneĭder [Shn1] which says

> *the series $\sum_{n=0}^{\infty} c_n w_n(x)$ converges on $(0,1)$ for every non-increasing sequence $\{c_n\}_{n=0}^{\infty}$ of reals converging to zero,*

we obtain that the interval $(0,1)$ belongs to $\Sigma^p(\mathfrak{W})$ for every $p \geq 1$. Thus, the family $\Sigma^p(\mathfrak{W})$ is uninteresting.

By theorems 18.1 and 18.2, $\Sigma^2(\mathfrak{R}) \subseteq \mathcal{L} \cap \mathcal{K}$ and $\Sigma^1(\mathfrak{R}) \subseteq \mathcal{K}$. Immediately we have

Theorem 18.3

(1) $\mathrm{non}(\Sigma^1(\mathfrak{R})) \leq \mathrm{non}(\mathcal{K})$,

(2) $\mathrm{cov}(\mathcal{K}) \leq \mathrm{cov}(\Sigma^1(\mathfrak{R}))$,

(3) $\mathrm{non}(\Sigma^2(\mathfrak{R})) \leq \min\{\mathrm{non}(\mathcal{L}), \mathrm{non}(\mathcal{K})\}$,

(4) $\max\{\mathrm{cov}(\mathcal{L}), \mathrm{cov}(\mathcal{K})\} \leq \mathrm{cov}(\Sigma^2(\mathfrak{R}))$.

The following result is a variation on a Rothberger's result [Rot] concerning measure and category.

Theorem 18.4

(1) $\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\Sigma^1(\{f_n\}_{n=0}^{\infty}))$,

(2) $\mathrm{non}(\mathcal{L}) \geq \mathrm{cov}(\Sigma^1(\{f_n\}_{n=0}^{\infty}))$.

PROOF. Fix a sequence $\{c_n\}_{n=0}^{\infty}$ in $\ell^2 \setminus \ell^1$, e.g. $c_n = 1/(n+1)$, and consider the set

$$A = \{(x,y) \in [0,1] \times [0,1] : \sum_{n=0}^{\infty} c_n f_n(x) r_n(y) \text{ converges}\}.$$

By Theorem 18.1, for every $x \in [0,1]$, the set

$$A_x = \{y \in [0,1] : (x,y) \in A\}$$

has Lebesgue measure 1, and for any non-dyadic $y \in [0,1]$, the set

$$A^y = \{x \in [0,1] : (x,y) \in A\}$$

is in $\Sigma^1(\{f_n\}_{n=0}^{\infty})$.

If $X \subseteq [0,1]$ with $|X| < \mathrm{cov}(\mathcal{L})$ then there is a non-dyadic real y such that $y \in A_x$ for every $x \in X$. Let $c'(n) = r_n(y)c(n)$. Then $\{c'(n)\}_{n=0}^{\infty} \in \ell^2 - \ell^1$ and $\sum_{n=0}^{\infty} c'_n f_n(x)$ converges on X. Hence $\mathrm{cov}(\mathcal{L}) \leq \mathrm{non}(\Sigma^1(\{f_n\}_{n=0}^{\infty}))$.

If Y is set of non-dyadic reals, $Y \notin \mathcal{L}$, then for each $x \in [0,1]$ we have $A_x \cap Y \neq \emptyset$; i.e., there exists a real $y \in Y$ such that $x \in A^y$. Hence the family $\{A^y : y \in Y\} \subseteq \Sigma^1(\{f_n\}_{n=0}^{\infty})$ is a covering family and therefore $\mathrm{cov}(\Sigma^1(\{f_n\}_{n=0}^{\infty})) \leq \mathrm{non}(\mathcal{L})$.

□

Theorem 18.5 *For $p \geq 1$,*

$$\mathcal{A}^{\mathfrak{R}} \subseteq \Sigma^p(\mathfrak{R}).$$

PROOF. Let L be an infinite subset of ω. We set

$$c(L,n) = \begin{cases} (-1)^k/\ln k, & \text{for } n = L(k),\ k > 1, \\ 0, & \text{for } n \in \omega \setminus L \text{ and for } n = L(0),\ n = L(1). \end{cases}$$

Since the series $\sum_{n=2}^{\infty} 1/(\ln n)^p$ diverges for any $p \geq 1$, $\sum_{n=2}^{\infty} |c(L,n)|^p = \infty$, and so the set

$$B_L = \{x \in [0,1] : \sum_{n=0}^{\infty} c(L,n) r_n(x) \text{ converges}\}$$

is in $\Sigma^p(\mathfrak{R})$. We show that

$$X_L \subseteq B_L.$$

Let $x \in X_L$. Then either $L \subseteq^* S_x$ or $L \subseteq^* \omega \setminus S_x$. Assume first that e.g. $L \subseteq^* S_x$. Then there is an n_0 such that for every $n \geq n_0$, $n \in L$, we have $r_n(x) = -1$. Therefore,

$$\sum_{n=L(n_0)}^{\infty} c(L,n) r_n(x) = -\sum_{n=L(n_0)}^{\infty} c(L,n)$$

and the series on the right side does converge. Thus, $x \in B_L$.

In the case $L \subseteq^* \omega \setminus S_x$ we obtain $r_n(x) = 1$ for all but finitely many n's (without loss of generality we can assume that x is not a dyadic real), and the result follows in the same way.

□

So, by Theorem 14.3, we obtain

Corollary 18.6 *For each* $p \geq 1$,

(1) $\mathrm{non}(\Sigma^p(\mathfrak{R})) \geq \mathfrak{s}$,

(2) $\mathrm{cov}(\Sigma^p(\mathfrak{R})) \leq \mathfrak{r}$.

The next results for the Rademacher system now follow from Theorems 18.3, 18.4 and Corollary 18.6.

Corollary 18.7

(a) *For* $p = 1$,

(1) $\max\{\mathfrak{s}, \mathrm{cov}(\mathcal{L})\} \leq \mathrm{non}(\Sigma^1(\mathfrak{R})) \leq \mathrm{non}(\mathcal{K})$,
(2) $\mathrm{cov}(\mathcal{K}) \leq \mathrm{cov}(\Sigma^1(\mathfrak{R})) \leq \min\{\mathfrak{r}, \mathrm{non}(\mathcal{L})\}$.

(b) *For* $1 < p < 2$,

(1) $\mathfrak{s} \leq \mathrm{non}(\Sigma^p(\mathfrak{R})) \leq \mathrm{non}(\mathcal{K})$,
(2) $\mathrm{cov}(\mathcal{K}) \leq \mathrm{cov}(\Sigma^1(\mathfrak{R})) \leq \mathfrak{r}$.

(c) *For* $p \geq 2$,

(1) $\mathfrak{s} \leq \mathrm{non}(\Sigma^p(\mathfrak{R})) \leq \min\{\mathrm{non}(\mathcal{K}), \mathrm{non}(\mathcal{L})\}$,
(2) $\max\{\mathrm{cov}(\mathcal{K}), \mathrm{cov}(\mathcal{L})\} \leq \mathrm{cov}(\Sigma^p(\mathfrak{R})) \leq \mathfrak{r}$.

19 Some open problems

For the sake of brevity, in this section we understand by a ***CTTS-family*** (= a family of classical trigonometric thin sets) any of the families $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{B}_0$, $\mathcal{A}$, $w\mathcal{D}$, $\mathcal{U}$.

In Section 11 we raised ten questions about a family of thin sets. For the CTTS-families, all, except one of them, were answered. So, we raise one unanswered and one refining question:

Problem 19.1

(1) *Is it consistent with ZFC that* $w\mathcal{D}$ *has a Borel basis?*

(2) *Are the expansions of Borel U-sets again U-sets?*

One can easily see that we have answered questions $Q1$, $Q3$–5, $Q7$–10 for the families $\mathrm{Prm}(\mathcal{F})$, where $\mathcal{F}$ is a CTTS-family, with the exceptions

Problem 19.2

(1) *Is every $\mathcal{N}$-permitted or $\mathcal{B}$-permitted or $w\mathcal{D}$-permitted set σ-porous?*

(2) *Is every countable set $\mathcal{U}$-permitted?*

For the remaining questions, we do not know the answers. So

Problem 19.3 *Let $\mathcal{F}$ be a CTTS-family.*

(1) *Does the family* $\mathrm{Prm}(\mathcal{F})$ *have a Borel basis?*

(2) *Does the family* $\mathrm{Prm}(\mathcal{F})$ *contain a perfect set?*

In connections with these problems it seems to us that the following holds true.

Conjecture 19.4 *No perfect set is $\mathcal{F}$-permitted for $\mathcal{F}$ being a CTTS-family.*

We raise three further questions.

$Q11$ Does every perfect set contain an uncountable subset belonging to $\mathcal{F}$?

$Q12$ Does every perfect set contain a subset of cardinality $\mathfrak{c}$ belonging to $\mathcal{F}$?

$Q13$ Does every perfect set contain a perfect subset belonging to $\mathcal{F}$?

Let us remark on the following fact. Let $\mathcal{F}$ be a family of thin sets with a Borel basis. If a perfect (or Borel) set P contains an uncountable subset $B \in \mathcal{F}$ then there exists a perfect subset P' of P in $\mathcal{F}$. Therefore, for a family with a Borel basis, the answers to questions $Q11$–$Q13$ are equivalent. For a CTTS-family, an affirmative answer to questions $Q11$–$Q13$ follows from theorems 6.5 and 8.3.

The property "to be a γ-set" is topologically invariant. Therefore, if there exists a γ-set of cardinality $\mathfrak{c}$, then every perfect set contains a γ-subset of cardinality $\mathfrak{c}$. Thus by the results of Section 13, it is consistent (even Martin's Axiom implies this) that the answer to question $Q12$ is affirmative for the families $\mathrm{Prm}(\mathcal{F})$, $\mathcal{F} = p\mathcal{D}, \mathcal{N}_0, \mathcal{N}, \mathcal{A}, w\mathcal{D}$. We do not know the answers in the general case.

Problem 19.5 *What are the answers to questions* $Q11$–13 *for* $\mathrm{Prm}(\mathcal{F})$, *where $\mathcal{F}$ is a CTTS-family?*

In Section 10 we collected some inclusions and non-inclusions between the CTTS-families and/or some families of small sets. To obtain the complete picture of relationships between all of them, we need to answer the following

Problem 19.6

(1) $\mathcal{B}_0 \subseteq \mathcal{H}$, $\mathcal{B}_0 \subseteq \mathcal{P}$?

(2) $a\mathcal{D} \subseteq \mathcal{H}$, $a\mathcal{D} \subseteq \mathcal{B}_0$, $a\mathcal{D} \subseteq \mathcal{B}$?

(3) $\mathcal{P}_\sigma \subseteq \mathcal{U}$?

(4) *Is every Borel (in particular closed) U-set σ-porous?*

In connection with the results of Section 12, we ask:

Problem 19.7 *Let $\mathcal{F}$ be a CTTS-family.*

(1) *Is* $\mathrm{Prm}(\mathcal{F})$ *a σ-ideal?*

(2) *Is there a convenient characterization of either of the cardinals* $\mathrm{non}(\mathcal{F})$ *and* $\mathrm{cov}(\mathcal{F})$*?*

(3) *Similarly, is there a convenient characterization of either of the cardinals* $\mathrm{non}(\mathrm{Prm}(\mathcal{F}))$ *and* $\mathrm{cov}(\mathrm{Prm}(\mathcal{F}))$*?*

Although we know answers to more than half of the thirteen questions $\mathcal{Q}1$–$\mathcal{Q}13$ about the families $\mathcal{A}^{\mathfrak{N}}$, $\mathcal{A}^{\mathfrak{W}}$, $\mathcal{U}^{\mathfrak{N}}$, $\mathcal{U}^{\mathfrak{W}}$, $\Sigma^p(\mathfrak{N})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$, we are far from being able to give complete answers as we did in the case of CTTS-families. Here are the questions we are not able to answer.

Problem 19.8

(1) *Is the family* $\Sigma^p(\mathfrak{N})$ *an ideal?*

(2) *Do the families* $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$ *have Borel bases?*

(3) *Is every set from* $\Sigma^p(\mathfrak{N})$ *σ-porous?*

(4) *Is the family* $\Sigma^p(\mathfrak{N})$ *closed under adding a point?*

(5) *Are the families* $\mathcal{A}^{\mathfrak{N}}$, $\mathcal{A}^{\mathfrak{W}}$, $\mathcal{U}^{\mathfrak{N}}$, $\mathcal{U}^{\mathfrak{W}}$, $\Sigma^p(\mathfrak{N})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$ *closed under shifts and expansions?*

(6) *Does every perfect set contain an uncountable subset belonging to* $\mathcal{U}^{\mathfrak{W}}$, $\Sigma^p(\mathfrak{N})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$*?*

(7) *Does every perfect set contain a subset of cardinality* $\mathfrak{c}$ *belonging to* $\mathcal{U}^{\mathfrak{W}}$, $\Sigma^p(\mathfrak{N})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$*?*

(8) *Does every perfect set contain a perfect subset belonging to* $\mathcal{U}^{\mathfrak{W}}$, $\Sigma^p(\mathfrak{N})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{N}})$, $\mathrm{Prm}(\mathcal{A}^{\mathfrak{W}})$*?*

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