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ON GENERALIZED STOCHASTIC CONVERGENCE

In what follows, let Iand Nbe the ideals of first category and Lebesgue measure zero sets in $\mathbb{R}$, respectively. Jdenotes an arbitrary ideal of subsets of $\mathbb{R}$. If a statement is true for all $x \in \mathbb{R}$, with the possible exception of a set of points contained in an ideal J, then the statement is said to be true J-a.e. For any set $A \subset \mathbb{R}$, its characteristic function is χ_A.

It is well-known that if A is a measurable set, then the following three statements are equivalent:

(A) a is a Lebesgue density point of A.

(B) $\chi_{n(A-a)\cap(-1,1)} \to \chi_{(-1,1)}$ in measure.

(C) For every increasing sequence $m_n \in \mathbb{N}$, there is a subsequence m_{n_p} such that $\chi_{m_{n_p}(A-a)\cap(-1,1)} \to \chi_{(-1,1)}$ N-a.e.

The equivalence of (A) with (C) was noted by W. Wilczyński in 1982 [2], who substituted the ideal Ifor Nin (C) to obtain the definition of I-density. This idea can be extended even more, to the case of an arbitrary ideal J[1].

Definition 1 The number a is a J-density point of $A \subset \mathbb{R}$ if for every sequence $m_n \in \mathbb{N}$, there is a subsequence m_{n_p} such that $\chi_{n_{n_p}(A-a)\cap(-1,1)} \to \chi_{(-1,1)}$ J-a.e.

Moving in parallel to the usual development of the density topology, we can define $\Phi_{\mathcal{J}}(A)$ to be the set of all Jdensity points of A and define the ordinary density topology to be

$$\mathcal{T}_{\mathcal{N}} = \{A \subset \mathbb{R} : A \subset \Phi_{\mathcal{N}}(A) \text{ and } A \text{ is measurable}\}.$$

Similarly, the I-density topology is

$$\mathcal{T}_{\mathcal{I}} = \{A \subset \mathbb{R} : A \subset \Phi_{\mathcal{I}}(A) \text{ and } A \text{ has the Baire property}\}.$$

Looking at the definitions of these two topologies, one is led to the following question.

Problem 1 If $\mathcal{J}$ is an arbitrary ideal in $\mathbb{R}$, find a property P such that

$$\mathcal{T}'_{\mathcal{J}} = \{A \subset \mathbb{R} : A \subset \Phi_{\mathcal{J}}(A) \text{ and } A \text{ has property } P\}$$

is well-behaved in the sense that the Lebesgue density and I-density topologies are well-behaved.

To give a sense of what can go wrong, note that it is known there is a nonmeasurable and non-Baire set $A \subset \mathbb{R}$ such that $\chi_{n(A-a)\cap(-1,1)} \to \chi_{(-1,1)}$ for all $a \in A$ [1]. Thus, if no additional condition P is imposed in Problem 1, this set is clopen in the topology resulting from any ideal J—even if Jis the empty ideal! In particular, $\mathcal{T}_{\mathcal{N}} \neq \mathcal{T}'_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{I}} \neq \mathcal{T}'_{\mathcal{I}}$. .

It is possible to strengthen Definition 1 in hopes of improving the situation. Instead of taking sequences of integers in Definition 1, we can substitute for m_n an arbitrary sequence of positive real numbers $t_n \uparrow \infty$ to obtain a density operator $\Psi_{\mathcal{J}}(A, \{t_n : n < \omega\})$ associated with each such sequence t_n and ideal J. Then a *strong* J-density operator can be defined as

$$\Psi_{\mathcal{J}}(A) = \bigcap_{\{t_n\}} \Psi_{\mathcal{J}}(A, \{t_n : n < \omega\}.$$

Even in this case, there are undesirable consequences, if additional assumptions are not placed upon the open sets. For example, if $\mathcal{C}$ is the ideal of countable subsets of $\mathbb{R}$, then there is a non-measurable, non-Baire set $A \subset \Psi_{\mathcal{C}}(A)$. Assuming the continuum hypothesis or Martin's axiom, there is a non-measurable and non-Baire set $A \subset \Psi_{\mathcal{I}}(A) \cap \Psi_{\mathcal{N}}(A)$.

Some additional examples and questions about these topologies are explored in [1].

References

[1] Krzysztof Ciesielski, Lee Larson, and Krzysztof Ciesielski. *$\mathcal{I}$-Density Continuous Functions.* Number 515 in Memoirs Series. Amer. Math. Soc., 1994.

[2] Władyslaw Wilczyński. A generalization of the density topology. *Real. Anal. Exchange*, 8(1):16–20, 1982–83.