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On some ideals of sets

The following question has been circulated among the mathematicians working in real analysis in Łódź: Which ideal generating the σ-ideal $\mathbb{L}$ of Lebesgue null sets on the real line $\mathbb{R}$ is a good analogue of the ideal NWD of nowhere dense sets generating the σ-ideal $\mathbb{K}$ of meager sets on $\mathbb{R}$? We show a general method yielding nice generators of $\mathbb{L}$. The results are derived from the forthcoming paper [2] joint with T. Świątkowski.

Assume that $\mathcal{I}$ is a given σ-ideal of subsets of $\mathbb{R}$ with the following properties:
(1) $\mathcal{I}$ contains all singletons;
(2) $U \notin \mathcal{I}$ for each nonempty open $U \subseteq \mathbb{R}$;
(3) $\mathcal{I}$ is invariant with respect to all functions of the form $x \mapsto ax + b$ where $a, b \in \mathbb{R}$;
(4) $\mathcal{I}$ is *Borel supported* (i.e. for each $A \in \mathcal{I}$ there is a Borel set $B \in \mathcal{I}$ such that $A \subseteq B$).

We say that an ideal $\mathcal{J} \subseteq \mathcal{I}$ *generates* $\mathcal{I}$ if each $A \in \mathcal{I}$ can be expressed as $\bigcup_{n\in\omega} A_n$ for some sequence $\{A_n\}_{n\in\omega} \subseteq \mathcal{J}$. We say that $\mathcal{J}$ is a *nice generator* of $\mathcal{I}$ if additionally the following conditions hold:
(1*) $\mathcal{J}$ contains all singletons;
(2*) $\mathcal{J}|\mathcal{U} \neq \mathcal{I}|\mathcal{U}$ for each nonempty open $U \subseteq \mathbb{R}$ (where $\mathcal{J}|\mathcal{U} = \{\mathcal{E} \subseteq \mathcal{U} : \mathcal{E} \in \mathcal{J}\}$, and similarly for $\mathcal{I}|\mathcal{U}$);
(3*) $\mathcal{J}$ is invariant with respect to all mappings $x \mapsto ax + b$, $a, b \in \mathbb{R}$;
(4*) $\mathcal{J}$ is Borel supported;
(5*) $\mathcal{J}$ is not a σ-ideal;
(6*) for each $E \subseteq \mathbb{R}$, the condition $E \cap U \in \mathcal{J}$ for any open bounded $U \subseteq \mathbb{R}$ implies $E \in \mathcal{J}$.

We can assume a more general situation where $\mathbb{R}$ is replaced by an uncountable Polish space X. Conditions (1),(2),(4),(1*),(2*),(4*),(5*) remain the same. The mappings $x \mapsto ax + b$ in (3) and (3*) can be replaced by another natural class of functions (for instance, by the respective algebraic operations, if X forms a metric group or a ring). Condition (6*) is the same but it becomes trivial if X is bounded. In particular, we will consider $X = 2^\omega$

(the Cantor space) which forms a metric group with the coordinatewise addition modulo 2.

The ideal (σ-ideal) of all nowhere dense (meager) sets in X will be denoted again by NWD ($\mathbb{K}$). We say that a σ-ideal $\mathcal{I}$ is *orthogonal* to $\mathbb{K}$ if there are sets $A \in \mathcal{I}$ and $B \in \mathbb{K}$ satisfying $A \cup B = X$.

Theorem 1 *[2] Assume that $\mathcal{I}$ and $\mathcal{I}_\infty$ are σ-ideals of subsets of X such that $\mathcal{I}_\infty \subseteq \mathcal{I}$ and $\mathcal{I}_\infty$ is orthogonal to $\mathbb{K}$. Then we have:*
(a) $\mathcal{I} = \{\mathcal{A} \cup \mathcal{B} : \mathcal{A} \in \mathcal{I}_\infty, \mathcal{B} \in \mathbb{K} \cap \mathcal{I}\}$;
(b) $\mathcal{J} = \{\mathcal{A} \cup \mathcal{B} : \mathcal{A} \in \mathcal{I}_\infty, \mathcal{B} \in NWD \cap \mathcal{I}\}$ forms an ideal generating $\mathcal{I}$;
(c) if each nonempty open set $U \subseteq X$ contains a set $D \in NWD \cap \mathcal{I} \setminus \mathcal{I}_\infty$ then $\mathcal{J}|\mathcal{U} \neq \mathcal{I}|\mathcal{U}$ for each nonempty open set $U \subseteq X$.

Remark. Assertion (c) states that $\mathcal{J}$ fulfils (2*). Conditions (1*),(3*) and (4*) for $\mathcal{J}$ hold provided that $\mathcal{I}$ and $\mathcal{I}_\infty$ fulfil (1),(3) and (4). The construction given in the proof of (c) guarantees that (5*) holds. Condition (6*) is always true for any σ-ideal and it holds for NWD. Thus, by (b), the ideal $\mathcal{J}$ satisfies (6*). Consequently, Theorem 1 produces nice generators of $\mathcal{I}$.

Examples. (i) Considear any regular Borel measure μ on $\mathbb{R}$. Then the family $\mathcal{I}_\infty$ of null sets with respect to μ forms a σ-ideal othogonal to $\mathbb{K}$. We want $\mathcal{I}_\infty$ strictly smaller than $\mathbb{L}$ in the sense given in (c). For instance, p-dimensional Hausdorff measure on $\mathbb{R}$ with $0 < p < 1$ is good since in each interval there exists a Cantor-type nowhere dense perfect set belonging to $\mathbb{L}$ but of positive p-dimensional Hausdorff measure.

(ii) Consider Mycielski σ-ideals on 2^ω defined in [3]. Note that a Mycielski σ-ideal is orthogonal to $\mathbb{K}$ and invariant in the Cantor group [3]. Additionally, for each Mycielski σ-ideal $\mathcal{M}$ there exists a Mycielski σ-ideal $\mathcal{M}_\infty \subseteq \mathcal{M}$ such that each open nonempty set $U \subseteq 2^\omega$ contains a set $D \in \mathrm{NWD} \cap \mathcal{M} \setminus \mathcal{M}_\infty$ [2]. Thus, by Theorem 1, for each Mycielski σ-ideal $\mathcal{M}$ there is a Mycielski σ-ideal $\mathcal{M}_\infty \subseteq \mathcal{M}$ such that $\{A \cup B : A \in \mathcal{M}_\infty, \mathcal{B} \in \mathbb{K} \cap \mathcal{M}\}$ is an ideal generating $\mathcal{M}$ and fulfilling conditions (1*)–(5*).

Example (i) shows that there are many nice generators of $\mathbb{L}$. Observe also that conditions (1*)–(6*) do not characterize NWD among the ideals generating $\mathbb{K}$ since the family $\{A \cup B : A \in \mathrm{NWD}, B \in \mathbb{K} \cap \mathbb{L}\}$ forms an ideal (greater than NWD) generating $\mathbb{K}$ and fulfilling (1*)–(6*). So, conditions (1*)–(6*) describe the whole class of ideals generating a given σ-ideal with properties (1)–(4). It would be interesting to choose in a reasonable way a unique canonical generator.

Problem. Is it possible to give a characterization of NWD such that its measure analogue yields a unique ideal generating $\mathbb{L}$?

Finally, let us mention another question concerning measure and category posed in [1].

Consider the σ-ideal

$$\mathcal{I} = \{\mathcal{E} \subseteq \mathbb{R}^{\epsilon} : \mathcal{E} \subseteq \mathcal{A}^{\epsilon} \cup \mathcal{B}^{\epsilon} \text{ for some } \mathcal{A} \in \mathbb{K}, \mathcal{B} \in \mathbb{L}\}$$

and let $\mathcal{B}$ denote the family of all Borel sets in $\mathbb{R}^2$. Is it true that any family $\mathcal{F} \subseteq \mathcal{B} \setminus \mathcal{I}$ must be countable?

Just after the Łódź conference M. Laczkovich solved that problem in negative.

References

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