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SOME GENERAL METHODS FOR SHOWING DERIVATIVES ARE IN B_1

Define a function $h : \mathbb{R} \to \mathbb{R}$ to be a parameter function if all the following are satisfied: (1) h is measurable; (2) $h(G)$ is measurable whenever G is measurable; and, (3) if A_n is a sequence of Borel sets such that $|A_n| \to 0$, then $|h^{-1}(A_n)| \to 0$.

Parameter functions such as these can be used to define a generalized derivative as follows. If $f : \mathbb{R} \to \mathbb{R}$, let

$$Q(x,t) = \frac{1}{g(t)} \sum_{i=1}^{n} a_i f(x + h_i(t))$$

where the a_i are constants, the h_i are parameter functions and g is an arbitrary nonzero function. $Q(x,t)$ is a generalized difference quotient, and it can be used to define a derivative as

$$f^*(x) = \lim_{t \to 0} Q(x,t)$$

whenever this limit exists. By taking appropriate one-sided limits, the one-sided versions of this derivative, f^{*+}, f^{*-} can be defined. Approximate versions, $f^*_{\text{ap}}, f^{*+}_{\text{ap}}, f^{*-}_{\text{ap}}$ are defined in the obvious way.

Theorem 1 ([1]) *If f is a measurable function such that f^{*+}_{ap} exists everywhere (even with infinite values), then $f^{*+}_{\text{ap}} \in B_1$.*

By choosing the parameter functions and constants in appropriate ways, it is possible to use this theorem to show that many common generalized derivatives are in B_1. For example, choosing $g(t) = 1$, $h(t) = t$ and $Q(x,t) = f(x+t)$, it follows that any right or left approximately continuous function is in B_1. Choosing $g(t) = 2t$, $h_i(t) = (-1)^{i-1}t$ for $i = 1, 2$ and $a_i = (-1)^{i-1}$ shows that the approximate symmetric derivative of a measurable function is in B_1.

Problem 1 What are the "proper" conditions on the parameter functions to extend this to the case of I-approximate limits?

Problem 2 Can this method be generalized to selective limits?

References

[1] Lee Larson. A method for showing generalized derivatives are in Baire class one. In *Classical Real Analysis*, volume 42 of *Contemporary Mathematics*, pages 87–95. Amer. Math. Soc., 1985.