Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 435–435

Clifford E. Weil, Mathematics Department, Michigan State University, East Lansing, MI 48824, email: weil@math.msu.edu

COMPOSITIONS WITH DERIVATIVES

The result of Theorem 1 of the other article written by me in this report can be interpreted as saying that the composition of a derivative with the squaring function need not be a derivative. The question then arises as to what functions ϕ have the property that $\phi \circ f \in \mathcal{D}$ for each $f \in \mathcal{D}$. The answer quite simply is the ϕ must be a linear function; that is, $\phi(y) = ay + b$ for some a, $b \in \mathbb{R}$. However if the question is posed not for all of $\mathcal{D}$, but for a subset of $\mathcal{D}$, the answer may not be as simple. The subsets sets to be considered are those that were central to one of the major theorem in the other article mentioned above. The definitions are repeated here.

Definition 1 *Let* $p \in (0, \infty)$ *and set*

$$C_p = \{f \in \mathcal{D} : \lim_{h \to 0} \left(\frac{1}{h} \int_x^{x+h} |f(t) - f(x)|^p \, dt \right)^{\frac{1}{p}} = 0 \text{ for all } x \in \mathbb{R}\}$$

and also set

$$B_p = \{f \in \mathcal{D} : \limsup_{h \to 0} \left(\frac{1}{h} \int_x^{x+h} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

What is obvious is that if $\phi \in \text{Lip}$, then $\phi \circ f \in C_p$ (resp. B_p) for every $f \in C_p$ (resp. B_p) at least for $p \geq 1$. It is equally easy to see that if $1 \leq p < q$ and if $\phi \in \text{Lip}_{p/q}$, then $\phi \circ f \in C_q$ (resp. B_q) for every $f \in C_p$ (resp. B_p). If these implications are reversible and what is the situation for $p < 1$ are the questions to be resolved.