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ON THE CAUCHY DIFFERENCE

If $f : \mathbb{R} \to \mathbb{R}$ is a function such that its *Cauchy difference*

$$f(x+y) - f(x) - f(y) \tag{1}$$

takes rational values only, then f is a sum of an additive function and a function with rational values only. However, there are functions $f : \mathbb{R} \to \mathbb{R}$ such that (1) takes *integer* values only and for each additive $a : \mathbb{R} \to \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $f(x) - a(x) \notin \mathbb{Z}$ (see, e.g., [1; Part 2]). But, if (1) as a function of two real variables is Lebesgue measurable and takes integer values only, then f is a sum of an additive function and a (Lebesgue measurable) function with integer values only [2]. It seems that J. G. van der Corput [6] was the first who noticed that under some regularity condition imposed on a function $f : \mathbb{R} \to \mathbb{R}$ such that (1) takes integer values only, there exists a real constant c such that $f(x) - cx \in \mathbb{Z}$ for every $x \in \mathbb{R}$. In [5] the theorem of van der Corput was extended for functions defined on real topological vector spaces as follows.

Theorem 1 *Suppose E is a real topological vector space. If the Cauchy difference* (1) *of a function $f : E \to \mathbb{R}$ takes integer values only and there exist non-void and open sets $U \subset E$ and $W \subset \mathbb{R}$ such that*

$$f(U) \cap (W + \mathbb{Z}) = \emptyset,$$

then there exists a $g \in E^*$ such that

$$f(x) - g(x) \in \mathbb{Z} \text{ for every } x \in E. \tag{2}$$

It has been shown in [4] that the theorem of van der Corput cannot be extended for functions taking values in $\mathbb{R}^2$ (with a replacement of $\mathbb{Z}$ by a discrete subgroup of $\mathbb{R}^2$).

Theorem 1 implies the following corollary.

Corollary 1 *Suppose E is a real topological vector space. If the Cauchy difference* (1) *of a function $f : E \to \mathbb{R}$ takes integer values only and there exists a non-void and open set $U \subset E$ and a $\gamma \in (0, \frac{1}{2})$ such that*

$$f(U) \subset (-\gamma, \gamma) + \mathbb{Z}, \tag{3}$$

then (2) *holds with a $g \in E^*$.*

Consider now the case where

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \text{ for all orthogonal } x, y \in E. \tag{4}$$

The following has been proved in [3].

Theorem 2 *Suppose E is a real inner product space of dimension at least* 2. *If a function $f : E \to \mathbb{R}$ satisfies* (4) *and there exists a neighbourhood $U \subset E$ of the origin such that* (3) *holds for some $\gamma \in (0, \frac{1}{4})$, then there exists a real constant c and a $g \in E^*$ such that*

$$f(x) - c\|x\|^2 - g(x) \in \mathbb{Z} \textit{ for every } x \in E.$$

It is an open problem whether in Theorem 2 the number γ can be taken from the interval $(0, \frac{1}{2})$.

An application of Theorem 2 to *orthogonally exponential functions*, i.e. to functions satisfying

$$\Phi(x+y) = \Phi(x)\Phi(y) \text{ for all orthogonal } x, y \in E, \tag{5}$$

gives (see [3]) what follows.

Corollary 2 *Suppose E is a real inner product space of dimension at least* 2. *If a function $\Phi : E \to \mathbb{C}$ satisfies* (5) *and there exists a neighbourhood $U \subset E$ of the origin and a positive real constant β such that*

$$|\Phi(x)| \le \beta \mathcal{R}(\Phi(x)) \textit{ for every } x \in U,$$

then either Φ vanishes on $E \setminus \{0\}$, or there exist additive functions $a : \mathbb{R} \to \mathbb{R}$ and $A : E \to \mathbb{R}$, a real constant c and a $g \in E^*$ such that

$$\Phi(x) = \exp(a(\|x\|^2) + A(x) + i(c\|x\|^2 + g(x))) \textit{ for every } x \in E.$$

References

[1] J.A. Baker, *On some mathematical characters*, Glasnik Mathematički 25 (45) (1990), 319-328.

[2] K. Baron, *A remark on the stability of the Cauchy equation*, Rocznik Naukowo-Dydaktyczny WSP w Krakowie. Prace Matematyczne 11 (1985), 7-12.

[3] K. Baron and G.L. Forti, *Orthogonality and additivity modulo* $\mathbb{Z}$, Results in Mathematics (to appear).

[4] K. Baron, A. Simon and P. Volkmann, *On the Cauchy equation modulo a subgroup*. Manuscript.

[5] K. Baron and P. Volkmann, *On a Theorem of van der Corput*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 61 (1991), 189-195.

[6] J.G. van der Corput, *Goniometrische functies gekarakteriseerd door een functionaalbetrekking*, Euclides 17 (1940), 55-75.