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ON THE SUMS AND THE PRODUCTS OF QUASI-CONTINUOUS FUNCTIONS

Throughout this paper we will consider only functions from $\mathbb{R}$ into $\mathbb{R}$.

We say that a function f is *quasi-continuous* (resp. *cliquish*) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open set $U \ni x$ we can find a non-empty open set $V \subset U$ such that $\omega(f, \{x\} \cup V) < \varepsilon$ (resp. $\omega(f, V) < \varepsilon$), where the symbol ω denotes the oscillation, i.e., $\omega(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$.

We say that f is *quasi-continuous* (resp. *cliquish*) if it is quasi-continuous (resp. cliquish) at each point $x \in \mathbb{R}$. Cliquish functions are also known as *pointwise discontinuous*.

The following properties are well-known and easy to prove.

- A function f is quasi-continuous iff for each $x \in \mathbb{R}$ there exists a sequence $x_1, x_2, \ldots \in \mathcal{C}(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$.
- A function f is cliquish iff $\mathcal{C}(f)$ is residual. In particular, every Baire one function is cliquish.

1 Sums

It is easy to show that the sum of two quasi-continuous functions need not be quasi-continuous. However, it must be cliquish. Z. Grande proved in [3] that each cliquish function can be written as the sum of four quasi-continuous functions. It is a result of J. Borsík that three functions are enough [1], and the beneath theorem, which generalizes also results of [7] and [4], gives a complete answer—just use this theorem for $g_1 = f$ and $g_2 = 0$.

Theorem 1 *Let $g_1, \ldots, g_k$ be cliquish and $\eta > 0$. There is a Lebesgue function α such that $g_i + \alpha$ is Darboux and quasi-continuous for $i \in \{1, \ldots, k\}$, $\mathcal{C}(\alpha) \supset \bigcap_{i=1}^{k} \mathcal{C}(g_i)$ and $|\alpha| < \sup\{\omega(g_i, x) : i \in \{1, \ldots, k\}, x \in \mathbb{R}\} + \eta$ on $\mathbb{R}$.*

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The results mentioned in this note which are not quoted are new and have been submitted for publication elsewhere.

(The symbol $\omega(g_i, x)$ stands for $\lim_{r\to 0+} \omega(f, \{y \in \mathbb{R} : \varrho(x, y) < r\})$.)

This theorem implies, in particular, that in case f is Borel measurable, then we can require the summands to be so, too. We can however ask whether the summands can be chosen even more regular, if f is so. E.g., we can consider semicontinuity or boundedness below.

The first question has an affirmative answer. It was proved by Z. Grande [5] in the bounded case, and by me in the general case, since $f = (f + \alpha) + (-\alpha)$.

Theorem 2 *Given an upper semicontinuous function f and an $\eta > 0$ we can find a non-negative, lower semicontinuous, quasi-continuous, Lebesgue function α such that $f + \alpha$ is Darboux upper semicontinuous and quasi-continuous, $\mathcal{C}(\alpha) \supset \mathcal{C}(f)$ and $|\alpha| \leq \sup\{\omega(f, x) : x \in \mathbb{R}\} + \eta$ on $\mathbb{R}$.*

The other question has negative answer. An easy computation shows that the function $f(x) = |x| + \chi_{\{0\}}(x)$ cannot be written as the sum of non-negative quasi-continuous functions. Analyzing this example one can prove that condition (B) below is indeed necessary. However sufficiency is not so evident.

Theorem 3 *For each $k > 1$ and each function $f \in \mathfrak{A}$ the following three conditions are equivalent:*

(A) there exist non-negative quasi-continuous functions $h_1, \ldots, h_k$ such that $f = h_1 + \ldots + h_k$,

(B) f is non-negative, cliquish and $\limsup_{y\to x, y\in\mathcal{C}(f)} f(y) \geq f(x)/k$ for $x \notin \mathcal{C}(f)$,

(C) there exist non-negative quasi-continuous functions $h_1, \ldots, h_k \in \mathfrak{A}$ such that $f = h_1 + \ldots + h_k$ and for $i \in \{1, \ldots, k\}$: $\mathcal{C}(h_i) \supset \mathcal{C}(f)$ and $h_i(x) > 0$ whenever $f(x) > 0$, $x \in \mathbb{R}$.

In this theorem $\mathfrak{A}$ stands for a family of functions which fulfills some extra conditions. In particular, it can be any of the following families: all functions, measurable functions, or Baire class α. I do not know, however, whether it can be the family of all approximately continuous functions or all derivatives.

This theorem implies the following corollary.

Corollary 4 *For each $k > 1$ and each function $f \in \mathfrak{A}$ the following three conditions are equivalent:*

(a) *there exist bounded below quasi-continuous functions $h_1, \ldots, h_k$ such that $f = h_1 + \ldots + h_k$,*

(b) *f is a bounded below cliquish function and*

$$\inf\left\{ \limsup_{y\to x, y\in\mathcal{C}(f)} f(y) - f(x)/k : x \notin \mathcal{C}(f)\right\} > -\infty,$$

(c) *there exist bounded below quasi-continuous functions* $h_1, \ldots, h_k \in \mathfrak{A}$ *such that* $f = h_1 + \ldots + h_k$ *and* $C(h_i) \supset C(f)$ *for* $i \in \{1, \ldots, k\}$.

As an application observe that the positive function $f = \sum_{n\in\mathbb{N}} n\chi_{\{n\}}$ does not satisfy condition (b) of this corollary.

2 Products

T. Natkaniec proved in [6] the following theorem.

Theorem 5 *A function f can be factored into a (finite) product of quasi-continuous functions iff f is cliquish and*

($\star$) *each of the sets* $f^{-1}\big((-\infty,0)\big)$, $f^{-1}(0)$ *and* $f^{-1}\big((0,\infty)\big)$ *is the union of an open set and a nowhere dense set.*

In the proof he uses eight (!) quasi-continuous functions. Again J. Borsík [2] showed that three functions are enough and my result gives a complete answer.

Theorem 6 *For each function* $f \in \mathfrak{A}$ *the following conditions are equivalent*:

a) *There is a* $k \in \mathbb{N}$ *and quasi-continuous functions* $h_1, \ldots, h_k$ *such that* $f = h_1 \cdot \ldots \cdot h_k$.

b) *There are quasi-continuous functions* $h_1, h_2 \in \mathfrak{A}$ *with* $f = h_1 \cdot h_2$.

c) *Function f is cliquish and it satisfies condition ($\star$) of Theorem 5.*

(The family $\mathfrak{A}$ can be any of those mentioned after Theorem 3.)

One can notice that in the above theorem we require neither preserving the points of continuity by the factors, nor their boundedness in case f is bounded. It turns out we <u>cannot</u> claim these requirements.

Theorem 7 *For each* $k > 1$ *and each function* $f \in \mathfrak{A}$ *the following three conditions are equivalent*:

(A) *there exist arbitrary bounded quasi-continuous functions* $h_1, \ldots, h_k$ *such that* $f = h_1 \cdot \ldots \cdot h_k$,

(B) *f is a bounded cliquish function, it satisfies condition ($\star$), and moreover,*

($\bullet$) *there exists an* $L > 0$ *such that for each* $x \notin C(f)$

$$\liminf_{y\to x,\, y\in C(f)} |f(y)| \le L \cdot \sqrt[k]{|f(x)|},$$

(C) *There exist bounded quasi-continuous functions* $h_1, \ldots, h_k \in \mathfrak{A}$ *such that* $f = h_1 \cdot \ldots \cdot h_k$.

(Necessity of condition ($\bullet$) follows by Corollary 4.)

References

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