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STRONG SEMICONTINUITY OF REAL FUNCTIONS

In various problems one encounters measurability of functions of several variables. Z. Grande [1] gave some theorems concerning measurability of real functions defined on the product of $\mathbb{R}$. We show more a general theorem whose proof entirely differs from Grande's and appears more direct.

Our proof is based on a theorem of Ślęzak [2] concerning classes of Baire set valued functions. To quote this theorem, some notation must first be introduced.

Let $(X, \mathcal{T}(\mathcal{X}))$ and (Z,T(Z)) denote two topological spaces and let $F : X \to Z$ denote a multifunction; i. e., $F(x)$ is a nonempty subset of Z for $x \in X$. Write $F^{-}(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$.

Let $\Sigma_\alpha(X)$ and $\Pi_\alpha(X)$ denote, respectively, the additive and multiplicative classes α, $\alpha < \Omega$, in the Borel classification of the subsets of X; i. e., $\Sigma_1(X) = \mathbf{F}_\sigma$, $\Pi_1(X) = \mathbf{G}_\delta$, $\Sigma_2(X) = \mathbf{G}_{\delta\sigma}$, $\Pi_2(X) = \mathbf{F}_{\sigma\delta}$, and so forth.

Theorem 1 ([2], Thm. 1) *Let $(X, \mathcal{T}(\mathcal{X}))$ be a perfectly normal topological space and let Z be a Polish space. Suppose that $F : X \to Z$ is a multifunction with closed values. Then the following conditions are equivalent: (1) F is of lower class α $(\alpha > 0)$; i. e., $F^{-}(G) \in \Sigma_\alpha(X)$ for every open $G \subset Z$; and, (2) there exist Borel α functions $f_n : X \to Z$, $n = 1, 2, \ldots$ such that for every $x \in X$ we have $F(x) = \mathrm{cl}\{f_n(x) : n \in \mathbb{N}\}$.*

Now we can prove the following theorem.

Theorem 2 *Let (X, d) be a metric space and let (Y, ρ) be a separable and complete metric space. Let $f : X \times Y \to \mathbb{R}$ be a function such that all its x-sections f_x are quasi-continuous and upper semi-continuous (Such functions are called strongly upper semi-continuous.) and all its y-sections f^y are B_1. Then the function f belongs to the upper class 2 in the Young classification; i. e., $f^{-1}(-\infty, r) \in \mathbf{G}_{\delta\sigma}$ for every $r \in \mathbb{R}$.*

PROOF. Let $S = \{s_n : n \in \mathbb{N}\} \subset Y$ be ρ-dense. For $(x,y) \in X \times Y$, by the strong upper semicontinuity of f_x, there exists an open set $U \subset Y$ such that $y \in \mathrm{cl}U$ and $\lim_{z \to y \wedge z \in U} f(x,z) = f(x,y)$. Therefore to each point $(x,y) \in X \times Y$, there corresponds a sequence $n \to s_n(x,y) \in S$ such that $\lim_{n\to\infty} s_n(x,y) = (x,y)$ and $\lim_{n\to\infty} f(x, s_n(x,y)) = f(x,y)$. Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of the rational numbers. For every $(n,m) \in \mathbb{N} \times \mathbb{N}$, let us define a family of functions $f_{nm} : X \to Y \times \mathbb{R}$, putting $f_{nm}(x) = (s_n(x,y), \min(q_m, f(x, s_n(x,y))))$. Clearly

$$f_{nm} \in B_1, \quad \forall (n,m) \in \mathbb{N} \times \mathbb{N}. \tag{1}$$

Let $H(x) = \{f_{nm}(x) : (n,m) \in \mathbb{N} \times \mathbb{N}\}$ for $x \in X$ and let us define a multifunction $F : X \to Y \times \mathbb{R}$ by the formula $F(x) = \{(y,r) \in Y \times \mathbb{R} : f(x,y) \geq r\} \subset Y \times \mathbb{R}$. Notice that

$$F(x) = \mathrm{cl}H(x). \tag{2}$$

According to (1) and (2) by Theorem 1 we have F is lower class 1; i. e., $F^-(G) \in \mathbf{F}_\sigma$ for every open set $G \subset Y \times \mathbb{R}$. Let $\mathrm{gr}F = \{(x,y,r) \in (X \times Y \times \mathbb{R} : (y,r) = F(x)\}$ denote the graph of F. Observe that $\mathrm{gr}F \in \mathbf{F}_{\sigma\delta}$ and for every $r \in \mathbb{R}$ and every r-section of the set $\mathrm{gr}F$

$$\{(x,y) : (x,y,r) \in \mathrm{gr}F\} = \{(x,y) : f(x,y) \geq r\} \in \mathbf{F}_{\sigma\delta}. \tag{3}$$

Let $r \in \mathbb{R}$. Now we have

$$\begin{aligned} f^{-1}(-\infty, r) &= \{(x,y) \in X \times Y : f(x,y) < r\} \\ &= X \times Y - \{(x,y) \in X \times Y : f(x,y) \geq r\} \end{aligned}$$

So, by (3) we have $f^{-1}(-\infty, r) \in \mathbf{G}_{\delta\sigma}$, and the theorem has been proved.

Theorem 2 is a generalization of [1, Theorem 5], and moreover, shows that the function f is in lower class 2. The measurability of f can be obtained after weakening the assumptions about y-sections of f.

References

[1] Z. Grande, *Quelques remarques sur la semi-continuité supérieure*, Fund. Math. 126(1985), pp. 1–13.

[2] W. Ślęzak, *Some contributions to the theory of Borel α selectors*, Problemy Matematyczne, 5/6(1986), pp. 69–82.