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Jack B. Brown, Department of Mathematics, Auburn University, AL 36849-5310, email: jbbrown@ducvax.auburn.edu.

RESTRICTION THEOREMS IN REAL ANALYSIS

We first discuss the classical restriction theorems, including (1) Lusin's 1912 Theorem [12] about continuous restrictions of Lebesgue measurable functions to sets of positive measure, (2) Blumberg's 1922 Theorem [2] about continuous restrictions of arbitrary functions to dense sets, and (3) Nikodym's 1929 Theorem [15] about continuous restrictions of Baire measurable functions to residual sets. We consider improvements or variations on these results such as Lusin's 1916 theorem [13] about derivative restrictions of Lebesgue measurable functions to sets of full measure and Brown's 1971 theorem [3] about pointwise discontinuous restrictions of arbitrary functions to uncountably dense sets.

We then turn our attention to consideration of differentiable and smooth restrictions of continuous functions. It was probably known in the 1930's or 40's that every continuous function agrees on an uncountable set with some differentiable function. It would have followed from Lebesgue's 1904 Differentiability Theorem [11], Jarník's 1923 Derivative Extension Theorem [9], and known results concerning nowhere monotone functions such as those given in 1940 by Minakshisundarum [14]. At about that time, Ulam asked [Scottish Book Problem 17.1] (see [17]) whether every continuous function agrees with some real analytic function on some uncountable set. Zahorski showed in 1947 [19] that the answer is no because there exists a C^∞ function which has only finite intersection with every real analytic function. The remaining questions concerning intersections of continuous functions and functions in smoother classes became known as the "Ulam-Zahorski Problem". We discuss contributions to the solution of this problem which have been made in the papers by Bruckner, Ceder, and Weiss (1969) [7]. Laczkovich (1984) [10], Agronsky, Bruckner, Laczkovich, and Preiss (1985) [1], Brown (1990) [4]. and Olevskiĭ (1994) [16], who gave the final solution to the problem. We also discuss Brown's [5] variation on a theorem of Olevskiĭ [16] and variations on theorems due to Federer [8] and Whitney [18] concerning intersections of Lipschitz, Hölder class, and smooth functions.

We conclude with a discussion of the papers of Brown and Prikry (1987) [6] and Brown (1992) [5], concerning continuous-, derivative-, and differentiable-restrictions of functions which are Borel-, Lebesgue-, universally-, Baire-, or Marczewski-measurable.

A list of 7 unsolved problems is given.

References

[1] S. Agronsky, A. M. Bruckner, M. Laczkovich, and D. Preiss, *Convexity conditions and intersections with smooth functions*, Trans. Amer. Math. Soc. 289 (1985), 659-677.

[2] H. Blumberg, *New properties of all real functions*, ibid. 24 (1922), 113-128.

[3] J. B. Brown, *Metric spaces in which a strengthened form of Blumberg's theorem holds*, Fund. Math. 71 (1971), 243-253.

[4] J. B. Brown, *Differentiable restrictions of real functions*, Proc. Amer. Math. Soc. 108 (1990), 391-398.

[5] J. B. Brown, *Continuous-, derivative-, and differentiable-restrictions of measurable functions*, Fund. Math. 141 (1992), 85-95.

[6] J. B. Brown and K. Prikry, *Variations on Lusin's theorem*, Trans. Amer. Math. Soc. 302 (1987), 77-86.

[7] A. M. Bruckner, J. G. Ceder and M. L. Weiss, *On the differentiability structure of real functions*, ibid. 142 (1969), 1-13.

[8] H. Federer, *Surface area (I), (II)*, ibid. 55 (1944), 420-456.

[9] V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable qui laisse intacte la dérivabilité de la fonction*, Bull Int. Acad. Sci Bohême (1923).

[10] M. Laczkovich, *Differentiable restrictions of continuous functions*, Acta Math. Hungar. 44 (1984), 355-360.

[11] H. Lebesgue, *Leçons sur l'intíntegration et la recherche des fonctions primitives*, Gauthier-Villars, Paris, (1904).

[12] N. Lusin, *Sur les propriétés des fonctions measurables*, C. R. Acad. Sci. Paris 154 (1912), 1688-1690.

[13] N. Lusin, *Sur la recherche des fonctions primitives*, ibid. 162 (1916), 975-978.

[14] S. Minakshisundaram, *On the roots of a continuous non-differentiable function*, J. Indian Math. Soc. 4 (1940), 31-33.

[15] O. Nikodym, *Sur la condition de Baire*, Bull. Internat. Acad. Polon. 1929, 591-198.

[16] A. Olevskiĭ, *Ulam-Zahorski problem on free interpolation by smooth functions*, Trans. Amer. Math. Soc. (1994), 713-727.

[17] S. Ulam, *A collection of mathematical problems*, Interscience, New York, 1960.

[18] H. Whitney, *On totally differentiable and smooth functions*, Pac. J. Math. 1 (1951), 143-159.

[19] Z. Zahorski, *Sur l'ensemble des points singuliére d'une fonction d'une variable réele admettand des dérivées de tous les ordres*, Fund. Math. 34 (1947), 183-245.