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CLASSIFICATIONS OF BOREL MEASURABLE FUNCTIONS

Let X be a subset of a Polish space. Given $0 < \alpha < \omega_1$ $\mathbf{\Sigma}^0_\alpha(X)$ stands for the αth additive class in the hierarchy of Borel sets on X. Let now $\alpha < \omega_1$. Let
$\mathbf{L}_\alpha(X) = \{f : X \to R : f^{-1}((a,\infty)) \in \mathbf{\Sigma}^0_{1+\alpha}(X)$ for each real $a\}$,
$\mathbf{U}_\alpha(X) = \{f : X \to R : f^{-1}((-\infty,a)) \in \mathbf{\Sigma}^0_{1+\alpha}(X)$ for each real $a\}$,
$\mathbf{B}_\alpha(X) = \mathbf{L}_\alpha(X) \cap \mathbf{U}_\alpha(X)$,
$\mathbf{S}_\alpha = \{l + u : l \in \mathbf{L}_\alpha(X),\ u \in \mathbf{U}_\alpha(X)\}$.

These classes, though with different numbering, can be also defined in the following way. We start by putting $\mathcal{B}_0(X) = \mathcal{L}_0(X) = \mathcal{U}_0(X) = \mathcal{S}_0(X)$ to be the class $C(X)$ of all continuous real-valued functions on X, and then continue inductively for $0 < \alpha < \omega_1$:

$\mathcal{B}_\alpha(X) = \{\lim_{n\to\infty} f_n$: f_n are pointwise convergent , $f_n \in \mathcal{B}_{\alpha_n}(X),\ \alpha_n < \alpha\}$,
$\mathcal{L}_\alpha(X) = \{\lim_{n\to\infty} f_n$: f_n are pointwise convergent , $f_1 \le f_2 \le \cdots,\ f_n \in \mathcal{L}_{\alpha_n}(X),\ \alpha_n < \alpha\}$,
$\mathcal{U}_\alpha(X) = \{\lim_{n\to\infty} f_n$: f_n are pointwise convergent , $f_1 \ge f_2 \ge \cdots,\ f_n \in \mathcal{U}_{\alpha_n}(X),\ \alpha_n < \alpha\}$,
$\mathcal{S}_\alpha(X) = \{\sum_{i=1}^\infty f_n : \sum_{i=1}^\infty \mid f_n(x) \mid < \infty$ for each $x \in X,\ f_n \in \mathcal{S}_{\alpha_n}(X),\ \alpha_n < \alpha\}$.
The Lebesgue-Hausdorff theorem implies that for $\alpha < \omega_0$: $\mathbf{B}_\alpha(X) = \mathcal{B}_\alpha(X)$, $\mathbf{L}_\alpha(X) = \mathcal{L}_{\alpha+1}(X)$, $\mathbf{U}_\alpha(X) = \mathcal{U}_{\alpha+1}(X)$, $\mathbf{S}_\alpha(X) = \mathcal{S}_{\alpha+1}(X)$; and for $\alpha \ge \omega_0$: $\mathbf{B}_{\alpha+1}(X) = \mathcal{B}_\alpha(X)$, $\mathbf{L}_\alpha(X) = \mathcal{L}_\alpha(X)$, $\mathbf{U}_\alpha(X) = \mathcal{U}_\alpha(X)$, $\mathbf{S}_\alpha(X) = \mathcal{S}_\alpha(X)$.

The classes $\mathcal{B}_\alpha(X)$, $\mathcal{L}_\alpha(X) \cup \mathcal{U}_\alpha(X)$, $\mathcal{S}_\alpha(X)$ form (resp.) Baire's, Young's and Sierpiński's classifications of Borel measurable real-valued functions on X.

We have the following diagram (where arrows stand for inclusions):

$$\begin{array}{ccccccc} & & \mathbf{U}_\alpha(X) & & & & \\ & \nearrow & & \searrow & & & \\ \cdots\ \mathbf{B}_\alpha(X) & & & & \mathbf{S}_\alpha(X) & \to & \mathbf{B}_{\alpha+1}(X)\ \cdots \\ & \searrow & & \nearrow & & & \\ & & \mathbf{L}_\alpha(X) & & & & \end{array}$$

Let us add that $\mathbf{L}_0(X)$ and $\mathbf{U}_0(X)$ form the classes of (lower and upper, resp.) semicontinuous functions on X.

The following three problems are related to the investigation of the structure of Borel measurable functions defined on Polish spaces.

Let Z be an uncountable Polish space.

Problem 1 (Lusin) *Does there exist a function $f : Z \to R$ which is Borel measurable such that f cannot be expressed as a sum $f = \bigcup_{n=1}^{\infty} f_n$ where $f_n \in C(dom(f_n))$?*

Problem 2 (Kempisty, [Ke]) $\mathbf{B}_{\alpha+1}(Z) \neq \mathbf{S}_\alpha(Z)$, $\alpha > 0$*?*

The result $\mathbf{B}_1(Z) \neq \mathbf{S}_0(Z)$ was known earlier and was shown by Mazurkiewicz in [Ma] and Sierpiński in [S] in the same volume of Fundamenta Mathematicae where Kempisty posed his problem.

Problem 3 (Lindenbaum, [Li] and [Li, corr]) *Characterize the following family of functions:*

$\Phi_\alpha = \{f : I \to R : f \circ g \in \mathbf{S}_\alpha(Z)$ *whenever* $g \in \mathbf{S}_\alpha(Z)$ *and* $rg(g) \subseteq I$, *I is any interval* $\subseteq R$ *(proper or not)*$\}$.

The last question remained open in [Li] as the theorem on the class Φ_α formulated there was not correct. That it did not have a correct proof was noticed by Lindenbaum himself in [Li, corr].

Problem 1 was solved positively in [Kie] but a number of more subtle and/or more general results were obtained in [AN], [La], [CM], [CMPS]. The last two papers used the universal functions approach to the problem. This approach enabled the author to solve positively Problem 2 in [Mo_1] and Problem 3 in [Mo_2] characterizing the class Φ_α as the class of functions satisfying locally the Lipschitz condition on their domains.

A convenient language to express the results obtained in the papers mentioned above involve cardinal coefficients which we shall now define. Let Z be an uncountable Polish space. Let $\mathcal{F} \subseteq {}^Z R$ and $\mathcal{G} \subseteq \bigcup\{{}^X R : X \subseteq Z\}$. Let

$$dec(\mathcal{F},\mathcal{G}) = min\{\kappa : \forall f \in \mathcal{F}\ \exists\{f_\alpha : \alpha < \kappa\} \subseteq \mathcal{G}\ (f = \bigcup_{\alpha<\kappa} f_\alpha)\}.$$

Let $\mathbf{RB}_\alpha(Z) = \bigcup\{\mathbf{B}_\alpha(X) : X \subseteq Z\}$ and we define analogously $\mathbf{RL}_\alpha(Z)$, $\mathbf{RU}_\alpha(Z)$ and $\mathbf{RS}_\alpha(Z)$.

In the solution of Problem 1 ([Kie], [AN], [La], [CM]) it was shown that

$$dec(\mathbf{B}_{\alpha+1}(Z),\ \mathbf{RB}_\alpha(Z)) > \aleph_0.$$

In [CMPS] this result was strengthened to the inequality

$$dec(\mathbf{B}_{\alpha+1}(Z),\ \mathbf{RL}_\alpha(Z) \cup \mathbf{RU}_\alpha(Z)) > \aleph_0.$$

This was improved in $[\mathrm{Mo}_1]$ to

$$dec(\mathbf{B}_{\alpha+1}(Z),\ \mathbf{RS}_\alpha(Z)) > \aleph_0.$$

which, of course, solved Problem 2.

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