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STABILITY ASPECTS OF DELTA CONVEXITY

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let D be a nonempty open and convex subset of X. A map $F : D \to Y$ is termed *delta-convex* provided that there exists a continuous convex functional $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\| y^* \| = 1$. If this is the case then F is called to be *controlled* by f or F is a delta-convex mapping with a *control function* f.

The following two results are due to L. Veselý and L. Zajiček (in: *Delta-convex mappings between Banach spaces and applications.* Dissertationes Math. 289, Polish Scientific Publishers, Warszawa, 1989).

A *continuous* function $F : D \to Y$ is a delta-convex mapping controlled by a *continuous* function $f : D \to \mathbb{R}$ if and only if the functional inequality

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \right\| \leq \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

is satisfied for all $x, y \in D$. Any delta-convex mapping is locally Lipschitzian.

A joint generalization of these two results reads as follows.

Theorem 1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let D be a nonempty open and convex subset of X. Suppose that functions $F : D \to Y$ and $f : D \to \mathbb{R}$ satisfy the inequality*

$$(*) \qquad \left\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \right\| \leq \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

whenever $x, y \in D$. If the function

$$D \ni x \longmapsto \| F(x) \| + f(x) \in \mathbb{R}$$

is upper bounded on a set $T \subset D$ whose $\mathbb{Q}$−convex hull $\mathrm{conv}_{\mathbb{Q}}(T)$ forms a second category Baire subset of X, then F is locally Lipschitzian; in particular, F is a delta-convex mapping controlled by f.

With the use of Christensen measurable mappings (a generalization of Haar measurability to the case of Abelian Polish topological groups) we get easily the following analogue of Theorem 1.

Theorem 2. *Let $(X, \|\cdot\|)$ be a real separable Banach space and let $(Y, \|\cdot\|)$ be a real separable normed linear space. If $D \subset X$ is a nonempty open and convex set and Christensen measurable functions $F : D \to Y$ and $f : D \to \mathbb{R}$ satisfy inequality $(*)$ for all $x, y \in D$, then F is locally Lipschitzian; in particular, F is a delta-convex mapping controlled by f.*

From Theorems 1 and 2 one obtains immediately the following

Corollary 1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear (resp. separable Banach) spaces and let $F : X \to Y$ be such that*

$$\| F(x+y) - F(x) - F(y) \| \leq \| x \| + \| y \| - \| x+y \|, \quad x, y \in X.$$

If F is bounded on a second category Baire subset of D (resp. F is Christensen measurable), then F is locally Lipschitzian and there exists an $M \in \mathbb{R}$ such that

$$\| F(x) \| \leq M \| x \|$$

for all $x \in X$.

The main Hyers & Ulam type stability result presented below establishes the existence of an affine mapping $A : D \to Y$ such that the difference $\| F(x) - A(x) \|$ is majorized by some function of $f(x)$, for all $x \in D$, whenever the pair (F, f) yields a suitable solution to the inequality $(*)$.

Theorem 3. *Let $(X, \|\cdot\|)$ be a real normed linear space and let D be a nonempty open and convex subset of X. Assume that $(Y, \|\cdot\|)$ is a real Banach space and functions $F : D \to Y$ and $f : D \to \mathbb{R}$ satisfy inequality $(*)$ for all $x, y \in D$. If there exists an $x_o \in D$ such that*

$$(**) \qquad \lim_{n\to\infty} 2^n \left[f\left(x_o + \frac{1}{2^n} x \right) - f(x_o) \right] = 0$$

for every $x \in X$, then there exists a map $A : D \to Y$ such that

$$A\left(\frac{x+y}{2}\right) = \frac{A(x) + A(y)}{2}, \qquad x, y \in D,$$

and

$$\| F(x) - A(x) \| \leq f(x) - f(x_o), \quad x \in D.$$

In some cases one may replace condition (**) by the assumption that the control function attains a local minimum at some point of its domain. Namely, we have the following

Theorem 4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let D be a nonempty open and convex subset of X. Assume that functions $F : D \to Y$ and $f : D \to \mathbb{R}$ satisfy inequality $(*)$ for all $x, y \in D$. If f has a local minimum at a point $x_o \in D$ and the space $(Y, \|\cdot\|)$ is either*

(i) *reflexive*

or

(ii) *has the Hahn-Banach extension property*

or

(iii) *forms a boundedly complete Banach lattice with a strong unit ,*
then there exists a constant $M \in \mathbb{R}$ and an affine mapping $A : D \to Y$ such that

$$\| F(x) - A(x) \| \leq M \left(f(x) - f(x_o) + \varphi_o(x_o - x) \right), \quad x \in D,$$

where $\varphi_o : X \to \mathbb{R}$ is defined by the formula

$$\varphi_o(x) := \lim_{n \to \infty} 2^n \left[f\left(x_o + \frac{1}{2^n} x \right) - f(x_o) \right] ;$$

in cases (i) *and* (ii) *one may take $M = 1$.*

If, moreover, f is bounded above in a neighborhood of x_o, then

$$\| F(x) - A(x) \| \leq c \left(f(x) - f(x_o) + \| x - x_o \| \right), \quad x \in D,$$

for some $c \in \mathbb{R}$.

Corollary 2. *Under the assumptions of* Theorem 4, *if both $F : D \to Y$ and $f : D \to \mathbb{R}$ are continuous then there exists a constant $c_o \in \mathbb{R}$ such that*

$$\| F(x) - F(x_o) \| \leq c_o \left(f(x) - f(x_o) + \| x - x_o \| \right), \quad x \in D.$$