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# ON STRONG QUASI-CONTINUITY

## 1 NOTATIONS:

$\mathcal{R}$ - the set of all reals;

$N$ - the set of all positive integers;

$\mu_e$ ($\mu$) - the outer Lebesgue measure (the Lebesgue measure) in $\mathcal{R}$;

$d_u(A,x) = \limsup_{h\to 0} \frac{mu_e(A\cap(x-h,x+h))}{2h}$ - the upper density of $A$ at $x$;

$d_l(A,x) = \liminf_{h\to 0} \frac{\mu_e(A\cap(x-h,x+h))}{2h}$ - the lower density of $A$ at $x$;

$x \in \mathcal{R}$ is called a density point of a set $A$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$;

$\mathcal{T}_d = \{A \subset \mathcal{R}; A$ is measurable and every point $x \in A$ is a density point of $A\}$ denotes the density topology;

$int(A)$ ($int_d(A)$) - the Euclidean interior (the interior in $\mathcal{T}_d$) of $A$;

$cl(A)$ - the closure of $A$;

$\mathcal{T}_{ae} = \{A \in \mathcal{T}_d; \mu(A - int(A)) = 0\}$.

## 2 DEFINITIONS:

A function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is called quasi-continuous (in short q.c.) [cliquish (in short c.q.)] at a point $x$ if for every open set $U$ containing $x$ and for every positive real $\eta$ there is a nonempty open set $V \subset U$ such that $|f(t) - f(x)| < \eta$ for all points $t \in V$ [$osc_V f < \eta$].;

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A function $f$ is said to be strongly quasi-continuous (in short s.q.c.) [strongly cliquish (in short s.c.q.)] at a point $x$ if for every set $A \in \mathcal{T}_d$ containing $x$ and for every positive real $\eta$ there is an open interval $I$ such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in A \cap I$ (and $osc_{A \cap I} f < \eta$].

**Observation.** In the definition of s.q.c. function (of s.c.q. function) we can write "...for every $F_\sigma - setA \in \mathcal{T}_d$ ..." and we obtain an equivalent definition.

## 3 PROPERTIES:

**Remark 1** *A function $f : \mathcal{R} \longrightarrow \mathcal{R}$ is s.q.c. at a point $x \in \mathcal{R}$ iff there is an open set $U$ such that $d_u(U, x) > 0$ and the restricted function $f/(U \cup \{x\})$ is continuous at $x$.*

Evidently, if a function $f$ is s.q.c. (s.c.q.) at $x$ then it is also q.c. (c.q.) at $x$.

**Remark 2** *Every s.q.c. function $f$ is almost everywhere (with respect to $\mu$) continuous.*

Now let

- $\mathcal{C}$ be the family of all continuous functions;
- $\mathcal{C}(\mathcal{T}_{ae})$ - the family of all a.e. continuous functions, i.e. continuous in the topology $\mathcal{T}_{ae}$.;
- $\mathcal{C}_{ae}$ - the family of all almost everywhere continuous functions;
- $\mathcal{C}_{sqc}$ - the family of all s.q.c. functions;
- $\mathcal{C}_{scq}$ - the family of all s.c.q. functions;
- $\mathcal{C}_{qc}$ - the family of all q.c. functions;
- $\mathcal{C}_{cq}$ - the family of all c.q. functions.

Moreover, if $\Phi$ is a family of functions then let:

- $\Phi + \Phi = \{f + g; f, g \in \Phi\}$;
- $\Phi\Phi = \{fg; f, g \in \Phi\}$.

## 4 SUMS, PRODUCTS, MAXIMAL ADDITIVE AND MULTIPLICATIVE FAMILIES:

**Theorem 1**
- *$\mathcal{C}_{sqc} + \mathcal{C}_{sqc} = \mathcal{C}(\mathcal{T}_{ae})$;*
- *if $f \in \mathcal{C}_{ae}$ and if $\mu(cl(f^{-1}(0)) - int(f^{-1}(0))) = 0$ then $f \in \mathcal{C}_{sqc}\mathcal{C}_{sqc}$;*
- *for every $f \in \mathcal{C}_{ae}$ then there are $c \in \mathcal{R}$ and functions $g, h \in \mathcal{C}_{sqc}$ such that $f = c + gh$;*
- *if $f$ is the product of a finite family of s.q.c. functions then $f$ satisfies the following condition (H) if $A \subset cl(f^{-1}(0)) - f^{-1}(0)$ is such that $d_l(f^{-1}(0), x) = 1$ for every $x \in A$ then $A$ is nowhere dense in $f^{-1}(0)$;*
- *$Max_{add}(\mathcal{C}_{sqc}) = \{f; f + g \in \mathcal{C}_{sqc}$ for all $g \in \mathcal{C}_{sqc}\} = \mathcal{C}(\mathcal{T}_{ae})$;*
- *$f \in Max_{mult}(\mathcal{C}_{sqc}) = \{f; fg \in \mathcal{C}_{sqc}$ for all $g \in \mathcal{C}_{sqc}\}$ if and only if $f \in \mathcal{C}_{sqc}$ and if $f$ is not a.e. continuous at a point $x$ then $f(x) = 0$ and $d_u(f^{-1}(0), x) > 0$.*

**Corollary 1** *Since for every $f \in \mathcal{C}_{cq}$ there is a homeomorphism $h$ such that $f \circ h \in \mathcal{C}_{ae}$, for every $f \in \mathcal{C}_{cq}$ there are a constant $c$ and functions $f_1, \ldots, f_4 \in \mathcal{C}_{qc}$ such that $f = f_1 + f_2$ and $f = c + f_3 f_4$.*

**Problems.** Let $D$ be the family of all Darboux functions. Describe $D\mathcal{C}_{sqc} + D\mathcal{C}_{sqc}$, $D\mathcal{C}_{sqc}D\mathcal{C}_{sqc}$, and $\mathcal{C}_{sqc}\mathcal{C}_{sqc}$.

## 5 LIMITS OF SEQUENCES:

By standard proofs we can show that the uniform and transfinite convergence conserve the strongly quasicontinuity. Since s.q.c. functions are almost everywhere continuous and q.c., for investigate the pointwise convergence we remind Mauldin's Theorem:

**Theorem 2** *$f \in B_\alpha(\mathcal{C}_{ae})$ ($B_\alpha(\mathcal{C}_{ae})$ denotes the class $\alpha$ in the Baire system generated by $\mathcal{C}_{ae}$, $\alpha > 0$) iff there are an $F_\sigma$-set $A$ of measure zero and a function $g \in B_\alpha(\mathcal{C})$ such that $\{x; f(x) \neq g(x)\} \subset A$.*

and my theorem:

**Theorem 3** *$B(\mathcal{C}_{qc}) = \mathcal{C}_{cq}$ ($B(\Phi)$ denotes $B_1(\Phi)$).*

**Theorem 4**
- *There is $f \in (B(\mathcal{C}_{ae}) \cap B(\mathcal{C}_{qc})) - B(\mathcal{C}_{sqc})$;*

- *There is $f \in \mathcal{C}_{scq} - B(\mathcal{C}_{ae})$;*
- *$B(\mathcal{C}_{sqc}) \subset \mathcal{C}_{scq}$;*
- *Suppose that for $f$ there is a Baire 1 function $g$ such that for every $\eta > 0$ and for every $x$ such that $|f(x)-g(x)| \geq \eta$ there is a nondegenerate closed interval $I(x)$ containing $x$ and such that $\mu(I(x) \cap cl(\{t; |f(t) - g(t)| \geq \eta\})) = 0$. Then $f \in B(\mathcal{C}_{sqc})$;*
- *For every $f \in \mathcal{C}_{ae}$ there are Darboux functions $f_n \in \mathcal{C}_{sqc}$, $n \in N$, such that $\forall_{\eta>0}\forall_{m\in N}\exists_{p\in N}\forall_x \min_{i\leq p} |f_{m+i}(x) - f(x)| < \eta$, i.e. $(f_n)_n$ quasi-uniformly converges to $f$.*

**Corollary 2** *$\mathcal{C}_{ae} \subset B(\mathcal{C}_{sqc})$.*

**Problem.** Describe $B(\mathcal{C}_{sqc})$.

# 6 LATTICES AND SUPERPOSITIONS:

A family $\mathcal{A}$ of functions $f : \mathcal{R} \longrightarrow \mathcal{R}$ is a lattice iff $\max(f, g) \in \mathcal{A}$ and $\min(f, g) \in \mathcal{A}$ for all $f, g \in \mathcal{A}$.

Let $C(f)$ ($C_{sqc}(f)$ or $C_q(f)$) respectively denote the sets of all continuity points of $f$ (the set of all points where $f$ is s.q.c. or the set of all quasi-continuity points of $f$) and let $D(f) = \mathcal{R} - C(f)$ ($D_{sqc}(f) = \mathcal{R} - C_{sqc}(f)$ or $D_q(f) = \mathcal{R} - C_q(f)$).

Since the family of all almost everywhere continuous functions is a lattice, we obtain by Grande-Natkaniec's Theorem that the lattice $\mathcal{L}(\mathcal{C}_{sqc})$ generated by $\mathcal{C}_{sqc}$ (i.e. the least lattice containing $\mathcal{C}_{sqc}$) is contained in the family of all almost everywhere continuous functions $f$ such that the sets $D_q(f)$ are nowhere dense.

Let $\mathcal{L}$ be the family of all functions $f : \mathcal{R} \longrightarrow \mathcal{R}$ such that for every nonempty set $A \in \mathcal{T}_d$ the set $A \cap D_{sqc}(f)$ is nowhere dense in $A$.

**Theorem 5** *Every function $f \in \mathcal{L}$ is almost everywhere continuous.*

**Remark 3** *There is a function $f \in \mathcal{L}$ such that $\mu(cl(D(f))) > 0$.*

**Theorem 6** *The family $\mathcal{L}$ is a lattice which contains all s.q.c. functions.*

**Remark 4** *Obviously, the lattice $\mathcal{L}$ is contained in the lattice of all almost everywhere continuous functions $f$ such that the set $D_q(f)$ is nowhere dense. However there are almost everywhere continuous functions $f$ with nowhere dense the sets $D_q(f)$ which are not in $\mathcal{L}$.*

**Theorem 7** *Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a function such that the set*

$$H(f) = cl(D_{sqc}(f) \cap int_d(cl(D_{sqc}(f))))$$

*is of measure zero. Then there are s.q.c. functions $f_1, \dots, f_9$ such that* $f = \min(\max(f_1, f_2, f_3), \max(f_4, f_5, f_6), \max(f_7, f_8, f_9))$.

**Remark 5** *From Theorem 3 it follows that if $\mu(cl(D_{sqc}(g))) = 0$ then $g \in \mathcal{L}(Q_s)$.*

**Example 1.** Let $C_n$, $n \in N$, be Cantor sets of positive measure such that $C_{n+1} \subset int_d(C_n)$ are nowhere dense in $C_n$ for $n \in N$. Enumerate all points $x$ such that there is an integer $n \in N$ for which $x \in C_n$ and $x$ is unilaterally isolated in $C_n$ in a sequence $(a_n)_n$ such that $a_n \neq a_m$ for $n \neq m$, $n, m \in N$. Then the function

$$f(x) = \begin{cases} 1/n & if \quad x = a_n, n \in N \\ 0 & otherwise \end{cases}$$

belongs to $\mathcal{L}$, but $\mu(H(f)) > 0$.

**Problem** Describe the lattice $\mathcal{L}(Q_s)$. Is true the equality $\mathcal{L}(Q_s) = \mathcal{L}$?

**Theorem 8**
$Max_{max}(\mathcal{C}_{sqc}) = \{f; \max(f, g) \in \mathcal{C}_{sqc}$ *for all* $g \in \mathcal{C}_{sqc}\} =$
$Max_{min}(\mathcal{C}_{sqc}) = \{f; \min(f, g) \in \mathcal{C}_{sqc}$ *for all* $g \in \mathcal{C}_{sqc}\} = \mathcal{C}(\mathcal{T}_{ae})$.

**Theorem 9** $Max_{comp}(\mathcal{C}_{sqc}) = \{f; f \circ g \in \mathcal{C}_{sqc}$ *for all* $g \in \mathcal{C}_{sqc}\} = \mathcal{C}$.