Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 394–395

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SOME THIN SETS OF REAL ANALYSIS

We consider cardinal characteristics of some families of thin sets related to trigonometric series and to Rademacher system. The terminology concerning thin sets is that of [Bu, Ka]. We shall denote by $\mathcal{PD}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$ the family of all pseudo Dirichlet, N_0-, N- and A-sets, respectively. The small uncountable cardinals $\mathfrak{m}$, $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{b}$ are defined and investigated in [Va].

If $\mathcal{F}$ is a set of subsets of a given set X, we define

$$\begin{aligned} \mathrm{non}(\mathcal{F}) &= \min\{|A|; A \subseteq X \ \&\ A \notin \mathcal{F}\}, \\ \mathrm{add}(\mathcal{F}) &= \min\{|\mathcal{G}|; \mathcal{G} \subseteq \mathcal{F} \ \&\ \bigcup \mathcal{G} \notin \mathcal{F}\}. \end{aligned}$$

We define the set of all "ensembles permis" for a given family $\mathcal{F}$ as follows:

$$\mathrm{Perm}(\mathcal{F}) = \{A \subseteq X; (\forall B \in \mathcal{F}) \mathcal{A} \cup \mathcal{B} \in \mathcal{F}\}.$$

The classical results say that

$$\mathrm{non}(\mathcal{N}) \geq \mathrm{non}(\mathcal{N}_0) \geq \aleph_1, \quad \mathrm{non}(\mathrm{Perm}(\mathcal{N})) \geq \aleph_1.$$

N. N. Kholshchevnikova 1985 proved that

$$\mathrm{non}(\mathcal{N}_0) \geq \mathfrak{m}.$$

Z. Bukovská 1990 showed that

$$\mathrm{non}(\mathcal{PD}) \geq \mathfrak{p}.$$

In 1992, N. N. Kholshchevnikova showed that

$$\mathrm{non}(\mathrm{Perm}(\mathcal{A})) \geq \mathfrak{m}.$$

Z. Bukovská and L. Bukovský 1993 proved that

$$\mathrm{non}(\mathrm{Perm}(\mathcal{N})) \geq \mathfrak{p}, \quad \mathrm{non}(\mathrm{Perm}(\mathcal{PD})) \geq \mathfrak{p}.$$

T. Bartoszyński and M. Scheepers 1993 improved all those results showing that

$$\mathrm{non}(\mathrm{Perm}(\mathcal{PD})) \geq \mathfrak{h}, \quad \mathrm{non}(\mathrm{Perm}(\mathcal{N})) \geq \mathfrak{t},$$
$$\mathrm{non}(\mathrm{Perm}(\mathcal{N}_0)) \geq \mathfrak{h}, \quad \mathrm{non}(\mathrm{Perm}(\mathcal{A})) \geq \mathfrak{s}.$$

Recently, L. Bukovský, N. N. Kholshchevnikova and M. Repický obtained better estimates

$$\mathrm{non}(\mathrm{Perm}(\mathcal{PD})) \geq \min\{\mathfrak{s}, \mathfrak{b}\}, \quad \mathrm{non}(\mathrm{Perm}(\mathcal{N}_0)) \geq \min\{\mathfrak{s}, \mathfrak{b}\}.$$

Moreover, they showed that

$$|A| < \mathfrak{b},\ A \in \mathcal{A} \quad \text{implies} \quad A \in \mathcal{PD}.$$

The Rademacher orthogonal system is defined by

$$r_n(x) = \mathrm{sgn}(\sin(2^n \pi x)) \text{ for } x \in [0,1].$$

We can introduce corresponding notions of thin sets for Rademacher system as follows. A set $A \subseteq [0,1]$ is called an $\mathrm{R}^{\mathrm{Rad}}$-set if there exists a sequence $a_n, n = 0,1,\ldots$ of reals such that $\sum_{n=0}^{\infty} a_n r_n(x)$ converges on A and $\sum_{n=0}^{\infty} |a_n| = \infty$. N. N. Kholshchevnikova 1993 showed that every countable set is an $\mathrm{R}^{\mathrm{Rad}}$-set and every $\mathrm{R}^{\mathrm{Rad}}$-set is meager.

A set $A \subseteq [0,1]$ is called an $\mathrm{A}^{\mathrm{Rad}}$-set if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\{r_{n_k}(x)\}_{k=0}^{\infty}$ converges for every $x \in A$. L. Bukovský, N. N. Kholshchevnikova and M. Repický 1994 prove that

$$\mathrm{non}(\mathcal{R}^{\mathrm{Rad}}) \geq \mathfrak{s}, \quad \mathrm{non}(\mathcal{A}^{\mathrm{Rad}}) = \mathfrak{s}, \quad \mathrm{add}(\mathrm{Perm}(\mathcal{A}^{\mathrm{Rad}})) \geq \mathfrak{h}.$$

References

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