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COMPACT SUBSETS OF THE BAIRE SPACE

Let ω^ω be the Baire space, infinite sequences of natural numbers with the product topology. In this topology a set $K \subset \omega^\omega$ is compact iff there exists a finite branching tree $T \subseteq \omega^{<\omega}$ such that

$$K = [T] = \{x \in \omega^\omega : \forall n \in \omega \; x \restriction n \in T\}.$$

Theorem 1 *If there exists a countable standard model of ZFC, then there exists M, a countable standard model of ZFC, $N \supseteq M$, a generic extension of M, and $T \in N$ a finite branching subtree of $\omega^{<\omega}$ with the properties that*

1. $\forall f \in [T] \cap N \; \exists g \in M \cap \omega^\omega \; f(n) < g(n)$ *for all but finitely many* $n \in \omega$ *and*
2. $\forall g \in M \cap \omega^\omega \; \exists f \in [T] \cap N \; g(n) < f(n)$ *for infinitely many* $n \in \omega$.

Since the elements of $[T]$ dominate the ground model and $[T]$ is finite branching, it must be that there exists $f \in N \cap \omega^\omega$ such that for every $g \in M \cap \omega^\omega$, $g(n) < f(n)$ for all but finitely many n. On the other hand every element of $[T]$ is weakly dominated by a ground model sequence. I don't know if in item 2 we can have the stronger condition that $g(n) < f(n)$ for all but finitely many $n \in \omega$.

This is related to Michael's problem [3] of whether there must be a Lindelöf space X such that $X \times \omega^\omega$ is not Lindelöf and M.E.Rudin's characterization of that problem [4], see also Alster [1] and Lawrence [2].

References

[1] K.Alster, *On the product of a Lindelöf space with the space of irrationals under Martin's Axiom*, Proceedings of the American Mathematical Society, 110(1990), 543–547.

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[4] M.E.Rudin, *An analysis of Michael's problem*, preprint 1993.