Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 387–391

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MULTIPLYING DERIVATIVES

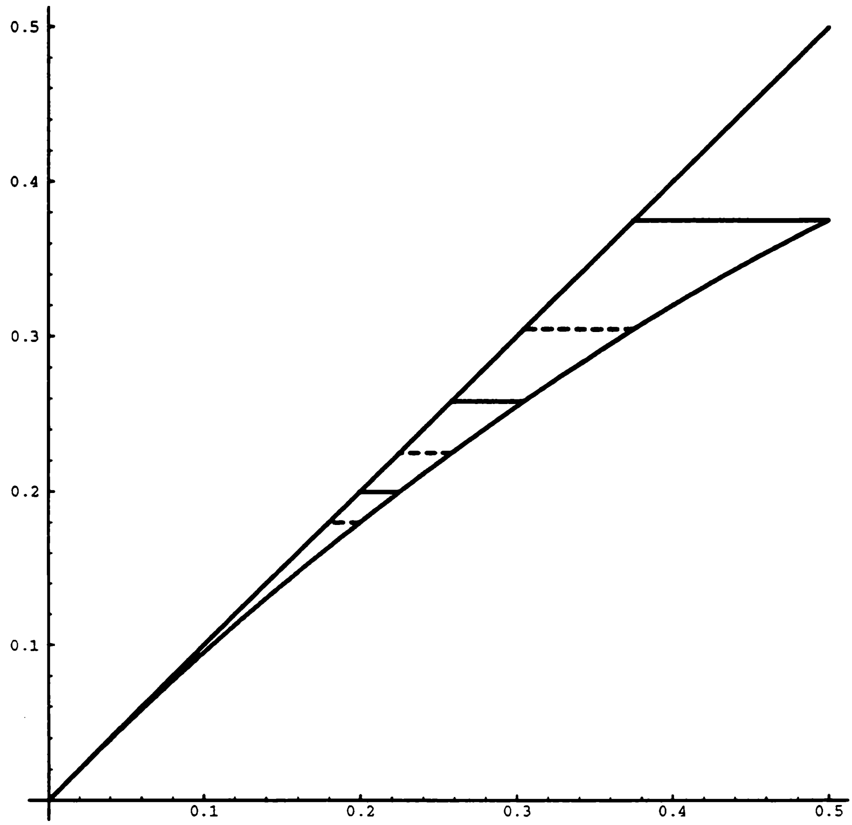
As a tribute to Jan Mařík, who died in January 1994, I would like to present some results from some of our joint work over the years. I will restrict myself to that part of our joint work concerned with multiplying derivatives. This is an appropriate topic to discuss in Poland since the first person to publish the fact that the product of two derivatives need not be a derivative was the Polish mathematician, W. Wilkosz in 1921.

Theorem 1 (Wilkosz) *The function*

$$f(x) = \begin{cases} \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a derivative, but the function f^2 is not a derivative.

The question then naturally arises as to what functions can be written at the product of two or more derivatives? For example can the product of two derivatives fail to have the Darboux property? The answer is yes. In fact the characteristic function of $\{0\}$ can be written as the product of two derivatives.



Continue constructing solid line segments and dotted ones as indicated. Then connect all of the solid ones with smooth curves that don't wander too far away from the given graphs and do the same thing for the dotted line segments. The resulting graphs will be graphs of two differentiable functions the product of whose derivatives is zero except at 0 where the product is 1.

The general case of the characteristic function of an arbitrary closed set is somewhat more difficult, but is a consequence of the first general theorem I would like to present. To do so some notation and definitions are needed that will be used throughout the talk.

Notation. Let $n \in \mathbb{N} \setminus 1$ and for each $i = 1, \ldots n$ let $\beta_i \in (0, \infty)$. Set $\beta = \sum_{i=1}^{n} \beta_i$.

The important objects for the statements of the first two theorems are the following sets.

Definition 1 *For any set $S \subset \mathbb{R}$ let*

$$\mathcal{D}(S) \;=\; \{f : S \to \mathbb{R} : f(x) = F'(x) \text{ for some } F : S \to \mathbb{R} \text{ and for each } x \in S\}$$

where the derivative is computed relative to S. Also let

$$\mathcal{P}(S) \;=\; \{f = \prod_{i=1}^{n} f_i^{\beta_i} : f_i \in \mathcal{D}(S) \text{ and } f_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}.$$

We will be concerned with just two cases: when S is an open set and when S is an interval which need not be open.

Theorem 2 *Let $G \subset \mathbb{R}$ be open, let $u \in \mathcal{P}(G)$ and let $v \in \mathcal{P}(\mathbb{R})$. Set*

$$w(x) = \begin{cases} u(x) & \text{if } x \in G \\ v(x) & \text{if } x \notin G. \end{cases}$$

Suppose $u \leq v$ on G. Then $w \in \mathcal{P}(G)$.

If we let v be the function identically 1 and let u be the function identically 0 on the complement G of a given closed set H, then the function w of the above theorem is the characteristic function of H. Clearly u and v satisfy the condition of the theorem with $n = 2$ and $\beta_1 = \beta_2 = 1$. Consequently w can be written as the product of two derivatives

Given that the characteristic function of a closed set is the product of two derivatives, what about the characteristic function of an open set? The next theorem has as a consequence that the characteristic function of a non-trivial open set can't be written as the product of any number of derivatives nor the product of an (fractional) powers of derivatives. The statement of the theorem requires some notation. We use the open interval $(0, \infty)$ but in fact any open interval could be used.

Definition 2 *Let $u : [0, \infty) \to \mathbb{R}$ be Lebesgue integrable on $[0, \infty)$. For δ, $\eta \in (0, \infty)$ let*

$$S_{\eta,\delta}(u) = \sup \left\{ \frac{1}{|I|} \int_I u : I \text{ is a subinterval of } [0, \delta) \text{ with } |I| \geq \eta \, dist(0, I) \right\}.$$

Note that $S_{\eta,\delta}(u)$ decreases as δ decreases to 0. So let

$$S_\eta(u) = \lim_{\delta \to 0+} S_{\eta,\delta}(u).$$

Also note that as η decreases to 0, $S_\eta(u)$ increases. So let

$$S(u) = \lim_{\eta \to 0^+} S_\eta(u).$$

Note that if $u \geq 1$ on $(0, \infty)$, then $S(u) \geq 1$.

Theorem 3 *Let $u : [0, \infty) \to \mathbb{R}$. Suppose $u|_{(0,\infty)} \in \mathcal{P}(0, \infty)$. Then $u \in \mathcal{P}([0, \infty))$ if and only if $u(0) \geq S^\beta(u^{\frac{1}{\beta}})$.*

Note that for the function identically 1 on the interval $(0, \infty)$ we have $S^\beta(u^{\frac{1}{\beta}}) = 1$. Consequently if $u \in \mathcal{P}([0, \infty))$, then $u(0) \geq 1$. Thus the characteristic function of $(0, \infty)$ isn't in $\mathcal{P}(\mathbb{R})$ for any choice of n or of the numbers β_i.

There is a curious fact that was discovered when working with these products. It is contained in the next theorem.

Theorem 4 *For each $i = 1, 2, \ldots, n$ let $f_i \in \mathcal{D}(S)$ with $f_i > 0$. Suppose $\prod_{i=1}^n f_i^{\beta_i}$ is approximately continuous at $x \in S$. Then each f_i is approximately continuous at x.*

For <u>sums</u> of powers of derivatives we have an analogous result. The role of the approximately continuous functions is played by the so-called Lebesgue functions. In what follows $\mathcal{D} = \mathcal{D}(\mathbb{R})$.

Definition 3 *Let*

$$\mathcal{L} = \{f : \mathbb{R} \to \mathbb{R} : \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0\}.$$

Theorem 5 *Let $p \in (0, \infty)$ and suppose $f_i \in \mathcal{D}$ for each $i = 1, 2, \ldots n$. Then $(\sum_{i=1}^n |f_i|^p)^{\frac{1}{p}} \in \mathcal{L}$ if and only is $f_i \in \mathcal{L}$ for each $i = 1, 2, \ldots, n$.*

Another class of functions important in the study of sum of powers of derivatives is the following class.

Definition 4 *Let*

$$\mathcal{M} = \{f \in \mathcal{D} : fg \in \mathcal{D} \text{ for each } g \in b\mathcal{A}\}$$

where

$$b\mathcal{A} = \{g : \mathbb{R} \to \mathbb{R} : g \text{ is bounded and approximately continuous on } \mathbb{R}\}.$$

This class is simply called the multipliers of the class $b\mathcal{A}$.

Theorem 6 *Let $p \in (0, \infty)$ and let $f_i \in \mathcal{M}$ for $i = 1, 2, \ldots, n$. Suppose that for each $x \in \mathbb{R}$*

$$\operatorname{ap}\liminf_{y \to x} \left(\sum_{i=1}^{n} f_i(y) \right)^{\frac{1}{p}} > 0.$$

Then $(\sum_{i=1}^{n} f_i)^{\frac{1}{p}} \in \mathcal{D}$ if and only if $(\sum_{i=1}^{n} f_i)^{\frac{1}{p}} \in \mathcal{M}$.

The study of multipliers of various classes of derivatives is the topic of our latest work. Returning to the definition of $\mathcal{L}$ interpret the conditions as requiring that the L^1-norm of $f - f(x)$ on the interval between x and $x + h$ where the measure of that interval is normalized to be 1, tends to 0 as h tends to 0. That is, f is continuous in the L^1-norm at x. The concept makes equally good sense for any $p \in (0, \infty)$

Definition 5 *Let $p \in (0, \infty)$ and set*

$$C_p = \{f \in \mathcal{D} : \lim_{h \to 0} \left(\frac{1}{h} \int_x^{x+h} |f(t) - f(x)|^p \, dt \right)^{\frac{1}{p}} = 0 \text{ for all } x \in \mathbb{R}\}$$

and also set

$$B_p = \{f \in \mathcal{D} : \limsup_{h \to 0} \left(\frac{1}{h} \int_x^{x+h} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The second space can be thought of as the space of all derivatives that are locally bounded in the L^p-norm.

Definition 6 *For any $S \subset \mathcal{D}$ let*

$$M(S) = \{f \in \mathcal{D} : fg \in \mathcal{D} \text{ for all } g \in S\}.$$

Theorem 7 *Let $p \in (1, \infty)$ and let let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$M(C_p) = B_{p'} \text{ and } M(B_p) = C_{p'}.$$