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OLD AND NEW SANDWICH THEOREMS

Let X be a set, $(Y, \leq)$ an ordered space, $\mathcal{F}$ and $\mathcal{G}$ two families of functions from X to Y such that $\mathcal{H} = \mathcal{F} \cap \mathcal{G} \neq \emptyset$. We assume that the classes $\mathcal{F}$ and $\mathcal{G}$ are mutually symmetric in some sense, e.g. if $Y = \mathbb{R}$ then we require $\mathcal{G} = -\mathcal{F} = \{-f : f \in \mathcal{F}\}$.

By *sandwich theorems* we mean results of the type: For each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ satisfying $f(x) \leq g(x)$, $x \in X$, there exists $h \in \mathcal{H}$ such that $f(x) \leq h(x) \leq g(x)$, $x \in X$. Note that such results are also called separation theorems.

It is known that under some additional assumptions, the sandwich theorem holds for the following pairs $(\mathcal{F}, \mathcal{G})$:

1. Superadditive and subadditive functions on Abelian semigroups.
2. Subadditive and superadditive set functions on rings.
3. Subharmonic and superharmonic functions.
4. Submartingales and supermartingales.
5. Upper and lower semicontinuous functions.
6. Concave and convex functions.

We shall deal with last two cases. In the semicontinuous case the following theorem is well known:

Theorem 1 *If X is a normal topological space, and $f, g : X \to \mathbb{R}$ are such that f is upper semicontinuous, g is lower semicontinuous and $f \leq g$, then there exists continuous $h : X \to \mathbb{R}$ satisfying $f \leq h \leq g$.*

E.Michael noticed that this result is a consequence of his selection theorem. In fact, required h is a continuous selector of the multifunction $\phi(x) = [f(x), g(x)]$, $x \in X$. Recently J.M.Borwein and M.Théra [2] have generalized Theorem 1 for the case when Y is a Banach space ordered by a convex cone. Their proof is based on the continuous selection theorem.

The above theorem has its analogue for multifunctions. Let ϕ be a multifunction from a topological space X to a metric space Z, i.e. the values of ϕ are nonempty subsets of Z. We say that ϕ is *Hausdorff-upper semicontinuous*, abbreviated to H-u.s.c., (respectively, *Hausdorff-lower semicontinuous*, abbrev. to H-l.s.c.) if for each $x_o \in X$ and $\epsilon > 0$ there is a neighborhood

U of x_o such that $\phi(x) \subset B(\phi(x_o), \epsilon)$ (respectively, $\phi(x_o) \subset B(\phi(x), \epsilon)$) for $x \in U$. For $A \subset Z$, $B(A, \epsilon)$ denotes open ϵ–ball around A. A multifunction ϕ is *Hausdorff-continuous* if it is H-l.s.c. and H-u.s.c.. By $\mathcal{F}(Z)$ we denote the family of all closed, convex, bounded and nonempty subsets of a normed vector space Z.

The following theorem holds:

Theorem 2 **([4])** *Suppose X is paracompact, Z is a Banach space, and $\phi, \psi : X \to \mathcal{F}(Z)$ are such that ϕ is H-u.s.c., ψ is H-l.s.c. and $\phi \subset \psi$. Then there exists Hausdorff-continuous $\chi : X \to \mathcal{F}(Z)$ satisfying $\phi \subset \chi \subset \psi$.*

The proof is based on the Rådström-Hörmander embedding theorem and on the Michael selection theorem. The assumption that ψ has bounded values is superfluous. If we additionally assume that ϕ is compact-valued, then we can obtain χ compact-valued too. Theorem 2 generalizes a result of S.M.Aseev [1]. Related sandwich theorem for multifunctions was given by F.S.De Blasi [3].

In the concave-convex case the following sandwich theorem is well known:

Theorem 3 *Let X be a convex subset of a real vector space V, and $f, g : X \to \mathbb{R}$. If X has nonempty relative algebraic interior, f is concave, g is convex and $f \leq g$, then there exists affine $h : V \to \mathbb{R}$ such that $f \leq h \leq g$.*

Under assumption that V is a topological vector space, one can look for continuous h. All such results are closely related to the Hahn-Banach extension theorem. In the seventies Theorem 3 was generalized for functions with values in an ordered vector space. We conjecture that it has an analogue for multifunctions.

Let X be a convex subset of a vector space, and Z another vector space. A multifunction ϕ from X to Z is *convex* if for all $x, y \in X$, $\lambda \in [0,1]$,

$$\phi(\lambda x + (1-\lambda)y) \supset \lambda\phi(x) + (1-\lambda)\phi(y).$$

Note that ϕ is convex iff its graph $gr\phi = \{(x,y) \in X \times Z : y \in \phi(x)\}$ is convex. In particular, any convex multifunction has convex values. We say that ϕ is *concave* if for all $x, y \in X$, $\lambda \in [0,1]$,

$$\phi(\lambda x + (1-\lambda)y) \subset \lambda\phi(x) + (1-\lambda)\phi(y).$$

Finally, a multifunction ϕ is called *affine* if it is convex and concave.

We conjecture that the following theorem holds: Suppose X is a convex subset of a real vector space , Z is a normed vector space, and $\phi, \psi : X \to \mathcal{F}(Z)$. If X has nonempty relative algebraic interior, ϕ is convex, ψ is concave and $\phi \subset \psi$, then there exists affine $\chi : X \to \mathcal{F}(Z)$ such that $\phi \subset \chi \subset \psi$. In the case $Z = \mathbb{R}$ the existence of such a multifunction χ is a consequence of Theorem 3.

References

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