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Krzysztof Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, e-mail: kcies@wvnvms.wvnet.edu

WHERE ANALYSIS, TOPOLOGY AND SET THEORY MEET: WHICH MATHEMATICAL OBJECTS CAN BE INTERESTING FOR TOPOLOGISTS?

The apparent success of topology as a branch of mathematics can be undoubtedly attributed to its wide applicability to the problems originated from the other parts of mathematics. Such applications include a large variety of results from the theory of real functions leading to the creation of functional analysis. All these early applications, however, were limited to the objects (like normed vector spaces) in which the topological structure was naturally existing. Can we apply topological methods for other natural "non-topological" mathematical objects? Can such "non-topological" objects be "made topological?" Here, we "make object topological" by finding a topological structure on a space (or spaces) involved from which the object under consideration can be defined in purely topological terms. For example, a family $\mathcal{G}$ of subsets of a set X can be made topological by finding a topology τ on X such that $\mathcal{G}$ is equal to either of: the topology τ, the family of all τ-closed sets, the family of all τ-Borel sets, the family of all G_δ sets, the ideal of all τ-nowhere dense sets, the σ-ideal of all τ-meager sets, etc. Similarly, a family $\mathcal{F}$ of functions from a set X into a set Y can be made topological by finding the topologies σ and τ on X and Y, respectively, such that the family $\mathcal{F}$ is equal to the family of: all continuous functions from (X,σ) into (Y,τ), all Baire one functions, all Borel functions, etc.

This note sketches the recent study in this direction. It is based on articles [2], [1], and [3] and consists on three corresponding parts.

Which classes of real functions can be topologized? In particular, can we topologize the following classes: Δ – of differentiable functions, $\mathcal{A}$ – of analytic functions, $\mathcal{P}$ – of polynomials, or $\mathcal{L}$ – of linear functions $f(x) = ax+b$?

Theorem 1 *Let $C^\infty \subset \mathcal{F} \subset \mathbb{R}^\mathbb{R}$ be $\circ$-closed. If $\mathcal{F}$ can be topologized then $\mathcal{F}$ is closed under max and min. In particular, classes C^∞ and Δ cannot be topologized.*

Theorem 2 *Let $\mathcal{A}$ = real or complex analytic functions. If GCH holds then there is a Hausdorff, connected topology τ s.t. $\mathcal{F} = \mathcal{C}(\tau_w, \tau)$ for every $\mathcal{F} \subset \mathcal{A}$ with Const $\subset \mathcal{F}$. Moreover, it is consistent that τ_w and τ are completely regular.*

Corollary 1 *It is consistent that there are completely regular, connected topologies $\tau_\mathcal{A}$, $\tau_\mathcal{P}$ and $\tau_\mathcal{L}$ s.t., $\mathcal{A} = \mathcal{C}(\tau_\mathcal{A}, \tau_\mathcal{A})$, $\mathcal{P} = \mathcal{C}(\tau_\mathcal{P}, \tau_\mathcal{P})$, and $\mathcal{L} = \mathcal{C}(\tau_\mathcal{L}, \tau_\mathcal{L})$.*

Harmonic functions from $\mathbb{R}^n$ into $\mathbb{R}$ can be topologized the same way.

Problem 1 *Can we prove Thm 2 without any additional set-theoretical assumptions? Can topologies from Thm 2 be normal? Lindelöf? hereditarily Lindelöf? compact? metrizable?*

There is, in ZFC, a Hausdorff, connected topology τ s.t. $\mathcal{F} = \mathcal{C}(\tau_w, \tau) \cap \mathcal{C}(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{O})$ for every $\mathcal{F} \subset \mathcal{A}$ with *Const* $\subset \mathcal{F}$.

Topologizing uniformly continuous functions. For which metric spaces X and Y the class $\mathcal{U}(X, Y)$ of all uniformly continuous functions from X to Y can be topologized as $\mathcal{C}(\sigma, \tau)$?

If X is compact or discrete then $\mathcal{U}(X, Y) = \mathcal{C}(X, Y)$ for any Y.

Theorem 3 *TFAE for any metric space (X, d)*

- $\mathcal{U}(X, \mathbb{R})$ *can be topologized;*
- $\mathcal{U}(X, Y)$ *can be topologized for some Y containing an arc;*
- $\mathcal{U}(X, Y)$ *can be topologized for any Y.*

Example 1 - *$\mathcal{U}(D, D)$ cannot be topologized for any dense $D \subset \mathbb{R}$.*

- *There are Bernstein sets $D, Y \subset \mathbb{R}$ s.t. $\mathcal{U}(D, Y)$ can be topologized. In particular, $\mathcal{U}(D, Y)$ can be topologized, while $\mathcal{U}(D, D)$ and $\mathcal{U}(D, \mathbb{R})$ cannot be.*

For a connected $C \subset \mathbb{R}, \mathcal{U}(C, C)$ can be topologized iff C is compact.

Theorem 4 *There is connected closed unbounded subset $X \subset \mathbb{R}^2$ such that $\mathcal{U}(X, X) = \mathcal{C}(\tau, \tau)$ for some connected Polish top. τ on X.*

Making ideals nowhere dense or meager. Let $\mathcal{J}$ be an ideal on a set X. Can we find a topology τ on X such that $\mathcal{J}$ is equal to the ideal of τ-meager sets? or τ-nowhere dense sets? How good such topologies can be? All ideals will be $\neq \mathcal{P}(X)$.

If $\mathcal{J}$ is a σ-ideal and τ makes $\mathcal{J}$ nowhere dense then τ makes $\mathcal{J}$ meager.

Theorem 5 *For every ideal $\mathcal{J}$ on X the family $\tau_{\mathcal{J}} = \mathcal{P}(X \setminus \bigcup \mathcal{J}) \cup \{X \setminus A : A \in \mathcal{J}\}$ is a T_0 top. making $\mathcal{J}$ nowhere dense. Moreover, $\tau_{\mathcal{J}}$ is T_1 if $\bigcup \mathcal{J} = X$.*

Since now, all topologies will be at least T_1. They will be T_2, if $\bigcup \mathcal{J} = X$.

Theorem 6 *There is metrizable top. making $\mathcal{P}(S)$ nowhere dense iff $|X \setminus S| \geq \omega$ and $|X \setminus S|^{\omega} \geq |S|$.*

Theorem 7 (GCH) *TFAE:*

(i) *there is compact T_2 making $\mathcal{P}(X)$ nowhere dense;*

(ii) *there is T_2 making $\mathcal{P}(X)$ nowhere dense;*

(iii) $|X \setminus S| \geq \omega$ *and* $|S| \leq 2^{2^{|X \setminus S|}}$.

Problem 2 *Can Thm 7 be proved on ZFC?*

Since now, all ideals will be σ-ideals containing all singletons.

An uncountable separable metric space is a *Lusin space* if $Meager(X) = [X]^{\leq \omega}$. It is known that CH implies that there is Lusin space, while MA+¬CH implies that there is no uncountable Hausdorff space X with $Meager(X) = [X]^{\leq \omega}$. Thus, Theorems 8 and 9 cannot be proved in ZFC.

Theorem 8 (CH) *For any σ-ideal $\mathcal{J}$ on a set X of cardinality continuum there is a Hausdorff top. making $\mathcal{J}$ meager.*

Theorem 9 *It is consistent with ZFC+GCH that for every σ-ideal $\mathcal{J}$ on $\mathbb{R}$ s.t.*

$$[\mathbb{R}]^{\leq \omega} \subset \mathcal{J} \subset Meager \cup Null$$

there is Hausdorff zero dimensional top. on $\mathbb{R}$ making $\mathcal{J}$ nowhere dense. Moreover, for the ideals with cofinality $\leq \omega_1$ the conclusion follows from CH.

Corollary 2 (CH) *Ideals of strong measure zero sets and of null sets can be made nowhere dense by Hausdorff zero dimensional top.*

Corollary 3 *It is consistent with ZFC+GCH that ideals of perfectly meager sets and universally measure zero sets can be made nowhere dense by Hausdorff zero dimensional top.*

Problem 3 *Can we find in ZFC a zero dimensional Hausdorff topology on $\mathbb{R}$ making meager sets nowhere dense?*

It is easy to see that under Martin's Axiom such a topology exists.

Problem 4 *Can topologies from Thm 9 be normal? compact? metrizable? In particular, can we have such topologies for the ideals of meager sets or null sets?*

References

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