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EVERY BOUNDED FUNCTION IS THE SUM OF THREE ALMOST CONTINUOUS BOUNDED FUNCTIONS*

In [1] it was asked whether every bounded function could be written as the sum of two almost continuous bounded functions and it was communicated that this question was open if "two" was replaced by "finite". In this note we show that "two" is indeed the *right* question; that is, we prove:

Theorem 1 *Suppose $f : [0,1] \to (-1,1)$ is arbitrary. Then there exist three almost continuous functions $g_i : [0,1] \to (-1,1)$, $i = 1,2,3$ such that $f = \sum_{i=1}^{3} g_i$.*

Proof. Let B_1 be a Bernstein set (a totally imperfect set whose complement is also totally imperfect), $B_2 = [0,1] \setminus B_1$ and enumerate the two sets as

$$B_1 = \{p_\alpha : \alpha < \omega_{\mathbf{c}}\} \text{ and } B_2 = \{q_\alpha : \alpha < \omega_{\mathbf{c}}\},$$

where $\omega_{\mathbf{c}}$ is the first ordinal of cardinality $2^{\aleph_0}$. Let $\mathcal{C} = \{C_\alpha : \alpha < \omega_{\mathbf{c}}\}$ be an enumeration of those closed subsets of $[0,1] \times (-1,1)$ whose projection on the x-axis (the Π_1 projection) has cardinality $\mathbf{c}$.

We first construct g_1 on B_1 so that $g_1|B_1$ (the graph of g_1 restricted to B_1) intersects C_ζ for all $\zeta < \omega_{\mathbf{c}}$. Let $\gamma(0) = \min\{\delta : p_0 \in \Pi_1(C_\delta)\}$. Suppose $\gamma(\beta)$ has been defined for all $\beta < \alpha$. Then define $\gamma(\alpha) = \min\{\delta : p_\alpha \in \Pi_1(C_\delta)$ and $\delta \neq \gamma(\beta)$ for all $\beta < \alpha\}$. Now for each $\alpha < \omega_{\mathbf{c}}$, let $g_1(p_\alpha)$ be such that $(p_\alpha, g_1(p_\alpha)) \in C_{\gamma(\alpha)}$. This completes the definition of g_1 on B_1. We next observe that $g_1|B_1$ intersects every C_ζ. To obtain a contradiction, assume

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that such is not the case and let ζ be the least ordinal such that $g_1|B_1$ does not intersect C_ζ. Since the cardinality of $\Pi_1(C_\zeta) \cap B_1 = \mathfrak{c}$ and γ is 1-1, we may choose α such that $\gamma(\alpha) > \zeta$ and $p_\alpha \in \Pi_1(C_\zeta)$. But this contradicts the minimality in the definition of $\gamma(\alpha)$.

We similarly construct g_2 on B_2 so that $g_2|B_2$ intersects every C_ζ. Now, let $g_1 = f$ on B_2, $g_2 = f$ on B_1 and let $g_3 = -g_1$ on B_1 and $g_3 = -g_2$ on B_2. Note that the graph of g_3 intersects every C_ζ and $f = \sum_{i=1}^{3} g_i$. It follows from the following lemma that g_i is almost continuous for $i = 1, 2, 3$. □

The following lemma appears in both [2] and [3, Theorem 1.2]; our proof is different, however, so we include it with this note.

Lemma 1 *Suppose $g : [0,1] \to (-1,1)$ is such that the graph of g intersects every C_ζ. Then, g is almost continuous.*

PROOF. Let $U \subset [0,1] \times (-1,1)$ be an open set containing the graph of g. Then, $(0, g(0)) \in U$ so

$$s \equiv \sup\{t : \exists \text{ a continuous } h : [0,t] \to (-1,1)\} > 0.$$

Assume $s < 1$. As $(s, g(s)) \in U$ there is a $\delta > 0$ such that $S \equiv (s-\delta, s+\delta) \times (g(s)-\delta, g(s)+\delta) \subset U$. Let $s - \delta/2 < t_* < s$ and let $h : [0, t_*] \to (-1,1)$ be continuous and $h \subset U$. If $h|_{[s-\delta,t_*]} \cap S \neq \emptyset$ then h can be extended continuously to $[0, s+\delta]$ contradicting the maximality of s. Hence, we may assume $h|_{[s-\delta,t_*]} \cap S = \emptyset$ and hence, $h|_{[s-\delta,t_*]}$ is either *above* or *below* S. For definiteness, suppose it is above and define

$$F = \{p \in [0,1] \times (-1,1) : \Pi_1(p) \in [s - \frac{\delta}{2}, t_*] \text{ and } \Pi_2(p) \in [g(s), h(\Pi_1(p))]\}.$$

Then $K \equiv F \cap U^c$ is closed and K does not intersect the graph of g. If $\Pi_1(K)$ is of cardinality $\mathfrak{c}$, then this leads to a contradiction as the graph of g intersects every C_ζ. Hence we may assume that $\Pi_1(K)$ is not of cardinality $\mathfrak{c}$ and choose an $x_0 \in [s - \frac{\delta}{2}, t_*]$ such that the line segment $[(x_0, g(s)), (x_0, h(x_0))]$ misses K. But then, there is an $\varepsilon > 0$ for which

$$\{(x,y) : x \in [x_o - \varepsilon, x_o + \varepsilon],\ y \in [g(s), h(x)]\} \subset U,$$

and again, h can be extended to $[0, s+\delta]$ which is a contradiction. Thus, we have shown that $s < 1$ leads to a contradiction. Hence, $s = 1$. Using an argument similar to the above, it can be easily shown that

$$1 \in \{t : \exists \text{ a continuous } h : [0,t] \to (-1,1)\}.$$

Hence, g is almost continuous. □

References

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