Real Analysis Exchange
Vol. 20(1), 1994/95, pp. 359–362

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DESCRIPTIVE MAPPING PROPERTIES OF TYPICAL CONTINUOUS FUNCTIONS

Abstract

We show that a typical continuous real function generates all analytic sets as image of G_δ-sets and all Borel sets as injective images of G_δ-sets.

In this note we answer a question posed by G. Petruska during the K&K-seminar in Salzburg, October 93.

Problem 1 *Is it true that for typical real valued continuous functions f on $[0,1]$ there always exists a G_δ-set whose f-image is not a Borel set?*

The affirmative answer is contained in the following stronger result.

Theorem 2 *For typical continuous f on $[0,1]$ and for any analytic set $A \subset f([0,1])$ there exists a G_δ-set M with $f(M) = A$. Moreover, in the same situation each Borel subset of $f([0,1])$ is an injective f-image of some G_δ-set.*

The proof is based on the standard idea to represent analytic sets as projections of G_δ plane sets and to use squarefilling Peano curves. However, since a typical curve is not squarefilling (e.g. the image has Hausdorff dimension one), we have to proceed more carefully. Before proving the Theorem we need two auxiliary results.

Lemma 3 *Let (X,ρ) be a metric space, $F \subset X$ closed and $M \subset X \setminus F$. If for each $\varepsilon > 0$ the set $\{x \in M \,;\, dist(x,F) > \varepsilon\}$ is G_δ in X, then M itself is a G_δ-set.*

Proof. We denote $U_0 = \{x \,;\, \operatorname{dist}(x,F) > 1\}$ and

$$U_i = \left\{x \,;\, \frac{1}{i+\frac{4}{3}} < \operatorname{dist}(x,F) < \frac{1}{i-\frac{1}{3}}\right\} \text{ for } i \geq 1.$$

Key Words: analytic and Borel sets, continuous and injective image, typical continuous function and Peano curves
Mathematical Reviews subject classification: Primary: 28A05, 26A99
Received by the editors April 12, 1994

Since $\bigcup_{i=0}^{\infty} U_i = X \setminus F$, we need only to show that $M \cap V$, $M \cap W$ and $M \cap U_0$ are G_δ-sets, where $V = \bigcup_{i=1}^{\infty} U_{2i}$ and $W = \bigcup_{i=1}^{\infty} U_{2i-1}$. Obviously, $U_0 \cap M$ is G_δ due to the assumption. Moreover, also for each $i \geq 1$ $M \cap U_i = U_i \cap \left\{x \in M \; ; \; \text{dist}(x, F) > \frac{1}{i+\frac{4}{3}}\right\}$ is a G_δ-set. Hence we find open sets $G_i^j \subset U_i$, $j \geq 1$, with $\bigcap_{j=1}^{\infty} G_i^j = M \cap U_i$. Since $U_i \cap U_{i'} = \emptyset$ for $|i - i'| \geq 2$, we see that both $V \cap M = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} G_{2i}^j = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} G_{2i}^j$ and $W \cap M = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} G_{2i-1}^j$ are G_δ-sets. □

The next statement showing that a typical function is essentially the x-coordinate of a squarefilling curve, is the "heart" of the paper.

Proposition 4 *Let D, E be dense subsets of $\mathbb{R}$. Then for typical continuous $f : [0,1] \to \mathbb{R}$ and for any $\varepsilon > 0$ there exists a compact set $C \subset [0,1]$ and continuous $g : [0,1] \to [0,1]$ such that*

$$(1) \quad \{(f(x), g(x)) \; ; \; x \in C\} = [\min f([0,1]) + \varepsilon, \max f([0,1]) - \varepsilon] \times [0,1]$$

and that the map

$$(2) \qquad t \to (f(t), g(t)) \text{ is injective on } C \setminus (f^{-1}(D) \cup g^{-1}(E)).$$

PROOF. We use the Banach-Mazur game, see [1], to show that for any $\varepsilon > 0$ the family $\mathcal{M}_\varepsilon$ of all $f \in \mathcal{C}([0,1], \mathbb{R})$ for which there is a $C \subset [0,1]$ compact and a $g \in \mathcal{C}([0,1],[0,1])$ fulfilling (1) and (2) is residual.

The following two simple observations (whose proofs are left to the reader) will be used during the game:

a) Let K, I be compact intervals, f continuous on K with $f(K) \supset I$ and let $\varepsilon > 0$ be given. Assume $\max I, \min I \in D$. Then for any N and $\delta > 0$ (we can always assume $\delta \ll \varepsilon$) there is a function $f' \in U(f, \varepsilon)$, a δ-fine division $\mathfrak{I}$ of I having all dividing points in D and there are mutually disjoint closed subintervals $K_{I',i}$ of $\text{int}\, K$ (for $I' \in \mathfrak{I}$ and $1 \leq i \leq N$), such that $f'(K_{I',i}) \supset U(I', \frac{\varepsilon}{4})$ and $|K_{I',i}| < \delta$ holds for any of these intervals.

b) Let $\{K_\alpha, \alpha \in A\}$ be a finite family of mutually disjoint closed intervals, and let g be a continuous function with $g(K_\alpha) \subset J$ for some interval J and all α. Whenever nonvoid $J_\alpha \subset J$ are selected for all $\alpha \in A$, then we find a continuous g' such that $\|g - g'\| \leq |J|$ and that always $g'(K_\alpha) \subset J_\alpha$.

Now, start the game and let us be given the answer $U_1 \subset \mathcal{C}([0,1], \mathbb{R})$ of the first move of player **A**. We find $f_1 \in U_1$, k_1 with $2^{-k_1} < \varepsilon$ and $U(f_1, 2^{-k_1}) \subset U_1$. We choose $m \in D \cap [\min f_1([0,1]), \min f_1([0,1]) + \frac{\varepsilon}{4})$, $M \in (\max f_1([0,1]) -$

$\frac{\varepsilon}{4}, \max f_1([0,1])]$, and (2^{-k_1})-fine divisions $\mathfrak{I}_1, \mathfrak{J}_1$ of $[m, M]$ and $[0,1]$ having all dividing points in D and E, respectively . According to the observations we can select mutually disjoint closed intervals $K_{I,J}$, $(I,J) \in \mathfrak{I}_1 \times \mathfrak{J}_1$, not longer than one and contained in $\operatorname{int}[0,1]$, and functions $f_2 \in U(f_1, 2^{-k_1-2})$, $g_2 \in \mathcal{C}([0,1],[0,1])$ such that for any pair $(I,J) \in \mathfrak{I}_1 \times \mathfrak{J}_1$ both $f_2(K_{I,J}) \supset U(I, 2^{-k_1-2})$ and $g_2(K_{I,J}) \subset J$ hold. Finally, we return our answer $U_2 = U_1 \cap U(f_2, 2^{-k_1-3})$ to player **A**.

Next, we consider the answer U_3 of **A**. Choose $k_2 > k_1 + 2$ and f_3 with $U(f_3, 2^{-k_2}) \subset U_3$. Since $f_3 \in U_2$, the inclusion $f_3(K_{I,J}) \supset U(I, 2^{-k_1-3})$ holds for any $(I,J) \in \mathfrak{I}_1 \times \mathfrak{J}_1$. We put $g_3 = g_2$ and find $\mathfrak{I}_2, \mathfrak{J}_2$ (2^{-k_2})-fine divisions refining $\mathfrak{I}_1$ resp. $\mathfrak{J}_1$ and again having all the endpoints of the corresponding subintervals in D and E, respectively. Oncemore applying the observations we can select $f_4 \in U(f_3, 2^{-k_2})$, and a map $g_4 \in U(g_3, 2^{-k_1})$ into $[0,1]$, and mutually disjoint closed intervals $K_{I,J}$ for $(I,J) \in \mathfrak{I}_2 \times \mathfrak{J}_2$ of length at most $1/2$ such that $K_{I,J} \subset \operatorname{int}(K_{I',J'})$ if $I \subset I' \in \mathfrak{I}_1$, $J \subset J' \in \mathfrak{J}_2$ and that $f_4(K_{I,J}) \supset U(I, 2^{-k_2-2})$, $g_4(K_{I,J}) \subset J$ for any pair from these second divisions. Then we return $U_4 = U_3 \cap U(f_4, 2^{-k_2-3})$.

We continue the game in this way. All we need to show is that for any $f \in \bigcap_{i=1}^{\infty} U_i$ there are g and C fulfilling (1), (2). The way we played the game ensures that $f_n \rightrightarrows f$ and also that the g_n's form a Cauchy sequence in $\mathcal{C}([\prime,\infty],[\prime,\infty])$. Denote its limit by g. It is obvious that $\max f \leq M + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2^{-k_1-3} < M + \frac{3\varepsilon}{4}$ and analogous $\min f > m - \frac{3\varepsilon}{4}$. We also put

$$\tilde{K}_{I,J} = K_{I,J} \cap (f,g)^{-1}(I \times J)$$

for any (I,J) in some $\mathfrak{I}_n \times \mathfrak{J}_n$, and define

$$\tilde{C} = \bigcap_{n \geq 1} \bigcup_{(I,J) \in \mathfrak{I}_n \times \mathfrak{J}_n} \tilde{K}_{I,J}.$$

We claim that always $\{(f(t), g(t))\ ;\ t \in \tilde{K}_{I,J}\} = I \times J$, by compactness and monotonicity this implies also (1). To verify the claim it suffices to show that for any $n \geq 1$, $x \in I$, $y \in J$ with $(I,J) \in \mathfrak{I}_n \times \mathfrak{J}_n$, and δ positive there is a $t \in K_{I,J}$ with $|f(t) - x| + |g(t) - y| < \delta$. For this purpose, we fix $N \geq n$ with $2^{-k_{N-1}+3} < \delta$ and $(I', J') \in \mathfrak{I}_N \times \mathfrak{J}_N$ such that $(x,y) \in I' \times J'$. Hence, $I' \times J' \subset I \times J$ and $K_{I',J'} \subset K_{I,J}$. Observe that $\|f - f_{2N}\| \leq 2^{-k_N-3}$, and $\|g - g_{2N}\| \leq \sum_{l=N}^{\infty} 2^{-k_{l-1}} \leq 2^{-k_{N-1}+1}$. Hence, for $K = K_{I',J'}$ we have $g(K) \subset U(g_{2N}(K), 2^{-k_{N-1}+1}) \subset U(J, 2^{-k_{N-1}+1}) \subset U(y, 2^{-k_N} + 2^{-k_{N-1}+1}) \subset U(y, \frac{\delta}{2})$. Moreover, we know that there is $t \in K$ satisfying $f_{2N}(t) = x$, hence $|f(t) - x| \leq 2^{-k_N-3} < \frac{\delta}{2}$ and $|f(t) - x| + |g(t) - y| < \delta$.

Now (2) follows easily. Indeed, let different $t, t' \in \tilde{C} \setminus (f^{-1}(D) \cup g^{-1}(E))$. be given. Since the maximal lenght of the $K_{I,J}$ goes to zero, we find an n and two different pairs $(I,J),(I',J') \in \mathfrak{J}_n \times \mathfrak{J}_n$ such that $t \in \tilde{K}_{I,J}$ and $t' \in \tilde{K}_{I',J'}$. Hence, $f(t) \in \operatorname{int} I$, $f(t') \in \operatorname{int} I'$, $g(t) \in \operatorname{int} J$, and $g(t') \in \operatorname{int} J'$. But this implies that $(f,g)(t) \neq (f,g)(t')$. Hence, we can choose C to be an appropriate subset of $\tilde{C}$. □

So we can turn to the

PROOF of Theorem 2. We denote $m = \min f([0,1])$, $M = \max f([0,1])$ and $F = f^{-1}(\{m, M\})$. Now let A be any analytic set contained in the range of f, obviously we can restrict to the case $A \subset (m, M)$. We decompose A for $k \in \mathbb{Z}$ into the analytic sets

$$A_k = \{t \in A \; ; \; \tan(\pi(\frac{f(t)-m}{M-m} - \frac{1}{2})) \in [k, k+1)\}.$$

Therefore, we can always choose plane G_δ-sets S_k contained in $A_k \times [0,1]$ with $A_k = \operatorname{proj}_x(S_k)$. Now Proposition 4 ensures the existence of a continuous map $g_k : [0,1] \to [0,1]$ such that the range of the map $h_k(t) = (f(t), g_k(t))$ contains the whole set S_k (and even its convex hull). Therefore, the G_δ-set $G_k = h_k^{-1}(S_k) \subset [0,1]$ is mapped onto A_k by f. So we are done if we prove that $G = \bigcup_{k \in \mathbb{Z}} G_k$ is a G_δ-set again. Obviously, $G \cap F = \emptyset$. Further, compactness easily implies that for any ε positive $\{x \in G \; ; \; \operatorname{dist}(x, F) > \varepsilon\}$ is an open subset of some finite union of G_k's, and consequently also an G_δ-set. Hence, an application of Lemma 3 finishes the first part of the proof.

In the second part, let $B \subset (m, M)$ be any Borel set. Obviously, it suffices to proof that $B^1 = B \setminus \mathbb{Q}$ as well as $B^2 = B \cap \mathbb{Q}$ are injective f images of some G_δ-sets. We define the B_k^1's and B_k^2's analogously to the A_k's. Since these sets form a partition of B, again it suffices to show that each single B_k^i is of the desired kind, i.e. is an injective f-image of some G_δ-set G_k^i. For $i = 1$ we set $D = \mathbb{Q}$, for $i = 2$ $D = \mathbb{R} \setminus \mathbb{Q}$ and $E = \mathbb{Q}$ in both cases. Because B_k^i is Borel, there is a relatively closed subset S_k^i of $[0,1] \times ([0,1] \setminus E)$ which is under the x-projection injectively mapped onto B_k^i. Obviously, S_k^i is of type G_δ in the plane and according to Proposition 4 there is a compact set $C \in [0,1]$ and a continuous $g : C \to [0,1]$ such that $(f,g)(C) \supset S_k^i$. Moreover, the G_δ-set $G_k^i = C \cap (f,g)^{-1}(S_k^i)$ is disjoint with $f^{-1}(D) \cup g^{-1}(E)$. ¿From (2) we know that (f,g) maps this set injectively onto S_k^i. Since $f = \operatorname{proj}_x \circ (f,g)$, we are done.

References

[1] J.Oxtoby, *The Banach-Mazur Game and Banach Category Theorem*, Contributions to the Theory of Games (ed. Dresher, Tucker, Wolfe), Ann. Math. Studies **39**, p. 159–163.