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ON DARBOUX BAIRE ONE FUNCTIONS

Abstract

In the paper I simplify the proof of the proposition which was the main tool in the work by Bruckner, Ceder, and Keston [2] concerning Darboux functions in the first class of Baire.

In 1968, Bruckner, Ceder, and Keston [2] proved several results concerning Darboux functions in the first class of Baire—the following three theorems in case $\alpha = 1$.

Theorem A *Let f be a Baire α function on an interval I. There exists a Darboux Baire α function g such that $f(x) = g(x)$ everywhere except on a first category set of measure zero.*

Theorem B *Let f be a Baire α function on an interval I. There exist two Darboux Baire α functions g and h such that $f(x) = g(x) + h(x)$ for all $x \in I$.*

Theorem C *Let f be a Baire $\alpha + 1$ function on an interval I. There exists a sequence of Darboux Baire α functions $\{f_n\}_{n=1}^{\infty}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in I$.*

(The above theorems for $\alpha > 1$ were proved earlier by other authors. For references see [2].) It was mentioned in [2] that the case of Baire class one needs a special treatment. However, contrary to the authors' opinion, this case can be handled surprisingly easily.

The tool in the proofs is the following proposition.

Proposition 1 *Let f be in the first class of Baire on an interval I and let E be a first category subset of I. There exists a Darboux Baire one function g such that $f = g$ except on a first category set of measure zero which is disjoint from E and such that the function $f - g$ is in the first class of Baire and has the Darboux property.*

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The original proof of the above proposition is rather complicated. (It takes a few pages.) The proof below is, in my opinion, much simpler. Instead of multiple induction, I use only the following lemma from [2].

Lemma 1 *Let C be a first category subset of an open interval (a,b) and let c and λ be real and positive extended real numbers respectively. There exists a Darboux Baire one function h defined on (a,b) such that $h = c$ except on a first category null set disjoint from C, and such that*

$$\liminf_{x\to a} h(x) = \liminf_{x\to b} h(x) = c - \lambda$$

and

$$\limsup_{x\to a} h(x) = \limsup_{x\to b} h(x) = c + \lambda.$$

PROOF OF PROPOSITION.
Denote by D the set of points of continuity of f. Set $\lambda_0 = \infty$, $A_0 = \emptyset$ and $\lambda_n = 2^{-n}$, $A_n = \{x \in I : \omega(f,x) \geq \lambda_n\}$ for $n \geq 1$ (ω denotes the oscillation, i.e., $\omega(f,x) = \lim_{\delta\to 0^+} \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in (x-\delta, x+\delta)\}$). Later we will use that each A_n is closed and nowhere dense.

Fix an $n \in \mathbb{N}$. Find a countable set $B_n \subset D$ such that $A_n \cup B_n$ is closed and each point of A_n is a point of bilateral accumulation of B_n. Use Lemma 1 for each component of $I \setminus (A_n \cup B_n)$ to construct a Darboux Baire one function $h_n : I \to [-\lambda_{n-1}, \lambda_{n-1}]$ such that $h_n = 0$ except on a first category null set which is disjoint from $E \cup (I \setminus D) \cup \bigcup_{i<n}\{x \in I : h_i(x) \neq 0\}$ and such that for each $a \in A_n \cup B_n$

$$\liminf_{x\to a^+} h_n(x) = \liminf_{x\to a^-} h_n(x) = -\lambda_{n-1}$$

and

$$\limsup_{x\to a^+} h_n(x) = \limsup_{x\to a^-} h_n(x) = \lambda_{n-1}.$$

Define function $h : I \to \mathbb{R}$ as $h = \sum_{n=1}^{\infty} h_n$. Since this series is uniformly convergent, h is a Darboux Baire one function. So the function g defined by $g = f - h$ is in the first class of Baire. Clearly $f = g$ except on a first category set of measure zero which is disjoint from E. To prove that g has the Darboux property, we will use the Young condition [1, Theorem 1.1, page 9], i.e., we will show that for each $x \in I$ there exist sequences $y_n \nearrow x$ and $z_n \searrow x$ such that $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(z_n) = g(x)$. Fix an $x \in I$. If $x \notin D$, then $h(x) = 0$ and $x \in A_k \setminus A_{k-1}$ for some $k \in \mathbb{N}$. Choose a sequence $\widetilde{y}_n \nearrow x$ of elements of B_k such that the limit $\widetilde{\lambda} = \lim_{n\to\infty} f(\widetilde{y}_n)$ does exist (not necessarily finite). Since $x \notin A_{k-1}$, $|\widetilde{\lambda} - f(x)| \leq \lambda_{k-1}$. Let $t_n \to 0$ be such that $|f(\widetilde{y}_n) - f(x) + t_n| < \lambda_{k-1}$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we can find a $y_n \in (\widetilde{y}_n, \widetilde{y}_{n+1})$ such that $h(y_n) = h_k(y_n) = f(\widetilde{y}_n) - f(x) + t_n$ and $|f(y_n) - f(\widetilde{y}_n)| < 1/n$. Then evidently $y_n \nearrow x$ and $\lim_{n\to\infty} g(y_n) = g(x)$. Now let $x \in D$. Select a sequence $y_n \nearrow x$ such that $\lim_{n\to\infty} h(y_n) = h(x)$. (Such a sequence exists since h is a Darboux Baire one function.) Then

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} f(y_n) - \lim_{n\to\infty} h(y_n) = f(x) - h(x) = g(x).$$

Similarly we can prove the existence of a sequence $z_n \searrow x$ such that $\lim_{n\to\infty} g(z_n) = g(x)$. This completes the proof of the proposition. □

Remark 1 In 1987, H. W. Pu and H. H. Pu [3] proved the following theorem, the proof of which was in fact a modified version of that of [2] and was as long as its prototype. We can adapt our proof to cover this case.

Theorem 1 *Let $\mathfrak{A}$ be a finite family of Baire one functions. Then there exists a Baire one function h such that $f + h$ is Darboux for every $f \in \mathfrak{A}$.*

SKETCH OF THE PROOF.
Define D to be the intersection of the sets of points of continuity of all $f \in \mathfrak{A}$. Set A_0 and $\lambda_0, \lambda_1, \ldots$ as above. Let $A_n = \bigcup_{f\in\mathfrak{A}}\{x \in I : \omega(f, x) \geq \lambda_n\}$ for $n \geq 1$. Construct function h as in the proof of Proposition 1 and use its arguments to show that $f + h$ is Darboux for every $f \in \mathfrak{A}$. □

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lect. Notes in Math., **659**, Springer Verlag, 1978.

[2] A. M. Bruckner, J. G. Ceder, and R. Keston, *Representations and approximations by Darboux functions in the first class of Baire*, Rev. Roumaine Math. Pures Appl. **13** (1968), p. 1247–1254.

[3] H. W. Pu and H. H. Pu, *On representations of Baire functions in a given family as sums of Baire Darboux functions with a common summand*, Časopis Pěst. Mat. **112** (1987), p. 320–326.