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ON ITERATIONS OF DARBOUX FUNCTIONS

Abstract

If $f : \mathbb{R} \to \mathbb{R}$ is weakly connected and for every $x \in \mathbb{R}$ there is an n_x with $f^{n_x}(x) = 1$ then $f(1) = 1$. The analogous result holds for any Darboux function f for which the set of all n_x is bounded above.

Let us establish some terminology to be used. Denote by $\mathbb{R}$ and $\mathbb{Q}$ the sets of reals and rationals, respectively, and by $\mathbf{I}$ the unit interval. We shall consider only real-valued functions of a real variable. No distinction is made between a function and its graph. Let:

$\mathcal{C}$ – denote the class of all continuous functions;

$\mathcal{C}onn$ – denote the class of all connected functions (i.e., functions whose graphs are connected in $\mathbb{R}^2$);

$\mathcal{C}onn_w$ – denote the class of all weakly connected functions. A function f is said to be *weakly connected* if for every interval J, $f|J$ can be separated by no continuous function $h : J \to \mathbb{R}$ (that is, $f \cap h = \emptyset$ implies that f intersects no more than one component of $(J \times \mathbb{R}) \setminus h$ [3]);

$\mathcal{D}$ – denote the class of Darboux functions, (i.e., functions which have the intermediate value property).

Obviously $\mathcal{C} \subset \mathcal{C}onn \subset \mathcal{C}onn_w \subset \mathcal{D}$.

Proposition 1 *For every $A \subset \mathbb{R}$ the following conditions are equivalent:*

(i) A is a bounded subset of $\mathbb{R}$;

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(ii) if f is a weakly connected function such that for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $f^n(x) \in A$, then $f(x) = x$ for some $x \in A$.

PROOF. $(i) \Rightarrow (ii)$. Let $a = \inf(A)$ and $b = \sup(A)$. Note that $f(b) \leq b$ and $f(a) \geq a$. Indeed, suppose $f(b) > b$. Since $f^n(b) \leq b$ for some $n > 1$, $f(x) \leq b$ for some $x > b$. So $f|(b, x)$ intersects the identity function and therefore $f(t) = t$ for some $t \notin A$. Consequently, $f^n(t) \notin A$ for all $n \in \mathbb{N}$, a contradiction. Analogously we can verify that $f(a) \geq a$. Since f is weakly connected, it intersects the identity and therefore $f(x) = x$ for some $x \in [a, b]$. Since $f(x) = x$ for all $n \in \mathbb{N}$, x is in A.

$(ii) \Rightarrow (i)$. Assume that A is unbounded above and $(a_n)_n$ is a strictly increasing sequence in A such that $\lim_n a_n = \infty$. Set $b_{n,m} = a_n + (1 - 2^{-m})(a_{n+1} - a_n)$ for all $n, m \in \mathbb{N}$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} a_1 & \text{if } x \leq a_0, \\ a_{n+1} & \text{if } x \in [a_n, b_{n,1}],\ n \in \mathbb{N}, \\ \text{linear} & \text{if } x \in [b_{n,1}, a_{n+1}],\ n \in \mathbb{N}. \end{cases}$$

Obviously, f is continuous. Fix $x \geq a_0$. Then $x \in [b_{n,m}, b_{n,m+1})$ for some $n, m \in \mathbb{N}$ and $f^{m+1}(x) = a_{n+m+1}$. Moreover, $f(x) > x$ for each $x \in \mathbb{R}$. □

Corollary 1 *If f is weakly connected and for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $f^n(x) = 1$, then $f(1) = 1$.*

The analogous result does not hold for all Darboux functions.

Proposition 2 *There exists a Darboux function f such that*

(1) $f(1) \neq 1$,

(2) for every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ with $f^n(x) = 1$.

PROOF. Let $\mathcal{B}$ be a countable base of $\mathbb{R}$. Arrange all elements of the set $\mathcal{B} \times \mathbb{R}$ in a sequence $(I_\alpha, y_\alpha)_{\alpha < 2^\omega}$. For every $\alpha < 2^\omega$ choose an

$$x_\alpha \in I_\alpha \setminus (\{x_\beta, y_\beta : \beta < \alpha\} \cup \{0, 1\} \cup \{y_\alpha\}).$$

Put

$$f(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0, \\ y_\alpha & \text{if } x = x_\alpha,\ \alpha < 2^\omega \\ 1 & \text{otherwise.} \end{cases}$$

Obviously f is Darboux and $f(1) \neq 1$. We shall verify the condition (2). Evidently, it is sufficient to consider only $x \in Z = \{x_\alpha : \alpha < 2^\omega\}$. Suppose that $f^n(x_\alpha) \neq 1$ for all $n \in \mathbb{N}$. Then $f^n(x_\alpha) \in Z$ for all n. Set $f^n(x_\alpha) = x_{\alpha_n}$. Then $x_{\alpha_1} = f(x_\alpha) = y_\alpha$, so $\alpha_1 < \alpha$ and $x_{\alpha_{n+1}} = f(x_{\alpha_n}) = y_{\alpha_n}$, so $\alpha_{n+1} < \alpha_n$ for each n. Consequently, $(\alpha_n)_n$ is a strictly decreasing sequence of ordinals, which is impossible. □

Remarks: 1. Since every Darboux function in the first class of Baire is connected [4], then no such function fulfills (1) and (2). We are unable to determine wheather or not there exists a Borel measurable Darboux function which fulfills (1) and (2).

2. Using the analogous arguments and standard methods of the construction of quasi-continuous functions possessing the Darboux property (see, e.g., [5]) we can construct a measurable quasi-continuous Darboux function which fulfills (1) and (2). Indeed, let C be the Cantor ternary set, $\mathcal{J}_n$ be the family of all components of the set $\mathbf{I} \setminus C$ of the n-th order (i.e. whose length is equal to 3^{-n}), $C_0 = C \setminus \{\mathrm{cl}(J) : J \in \mathcal{J}_n \text{ for some } n \in \mathbb{N}\}$. Arrange all rationals in a sequence $(t_n)_n$. Let $(q_n)_n$ and $(k_n)_n$ be sequences of rationals and of positive integers, respectively, defined as follows:

(1) $q_1 = t_1$ and $k_1 = 2$;

(2)
$$q_{n+1} = \begin{cases} 1 & \text{if } \{q_1, \ldots, q_n\} \cap \bigcup \mathcal{J}_{n+1} \neq \emptyset, \\ t_{k_n} & \text{otherwise;} \end{cases}$$

(3) $k_{n+1} = \min\{k : t_k \neq q_i \text{ for } i = 1, \ldots, n+1\}$.

Observe that for every $q \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ with $q = q_n$. Let $\mathcal{B}$ denote a countable base of C_0. Arrange all elements of the set $\mathcal{B} \times \mathbb{R}$ in a sequence $(J_\alpha, y_\alpha)_{\alpha < 2^\omega}$. For every $\alpha < 2^\omega$ choose an

$$x_\alpha \in J_\alpha \setminus (\mathbb{Q} \cup \{x_\beta, y_\beta : \beta < \alpha\} \cup \{y_\alpha\})$$

and put

$$f(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0, \\ y_\alpha & \text{for } x = x_\alpha,\ \alpha < 2^\omega, \\ q_n & \text{for } x \in \bigcup\{\mathrm{cl}(J) : J \in \mathcal{J}_n,\ q_n \notin J\},\ n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then f has all desired properties.

Let $\mathcal{I}$ be an ideal of boundary subsets of $\mathbb{R}$. We shall say that a function f is *$\mathcal{I}$-conservative* whenever $f(A) \in \mathcal{I}$ for all $A \in \mathcal{I}$.

Proposition 3 *Assume that f is an $\mathcal{I}$-conservative Darboux function, $A \in \mathcal{I}$, $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$ there exists an $i \leq n$ with $f^i(x) \in A$. Then there exists exactly one $a \in A$ such that $f(a) = a$.*

PROOF. We can assume that $n = \min\{k \in \mathbb{N} : \forall(x \in \mathbb{R})\, \exists(i \leq k)\, f^i(x) \in A\}$ and $A \subset f(\mathbb{R})$. Put $A_0 = A$ and $A_i = \{x : f^i(x) \in A\}$ for $i = 1, \dots, n$. Note that $\mathbb{R} = \bigcup_{i=1}^n A_i$, $f(A_i) \subset A_{i-1}$ for $i = 1, \dots, n$ and $A_n \neq \emptyset$. Since f is Darboux, f^i is Darboux for every i. Hence $J_i = f^i(\mathbb{R})$ is an interval for $i = 1, \dots, n$. Observe that $J_1 = f(\mathbb{R}) \subset \bigcup_{i=0}^{n-1} A_i$ and $A \subset J_1$; $J_2 = f(J_1) \subset f(A) \cup \bigcup_{i=0}^{n-2} A_i$ and $f(A) \subset J_2$; ... ; $J_n = f(J_{n-1}) \subset (A \cup f(A) \dots f^{n-1}(A))$ and $f^{n-1}(A) \subset J_n$. Since f is $\mathcal{I}$-conservative, J_n is a boundary set and consequently, it is a singleton. Let $J_n = \{a_0\}$. Clearly, $A \cap J_n \neq \emptyset$, so $a_0 \in A$ and $f^{n-1}(a_0) = a_0$. Hence $a_0 \in J_{n-1}$, $f(a_0) \in J_n$ and consequently, $f(a_0) = a_0$.

Now suppose $f(a) = a$ for some $a \in A$. Then $a = f^n(a) = a_0$. □

Corollary 2 *Assume that f is a Darboux function, $A \subset \mathbb{R}$,* card$(A) < 2^\omega$*, $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$ there exists an $i \leq n$ such that $f^i(x) \in A$. Then $f(a) = a$ for exactly one $a \in A$.*

Let $\mathcal{N}$ denote the ideal of all Lebesgue measure zero sets. We say that a function f satisfies the *Lusin condition* (N) iff f is $\mathcal{N}$-conservative. It is well-known (and easy to obtain) that every Lipschitz function satisfies the condition (N). (f is a *Lipschitz* function if there exists an $L > 0$ with $|f(x_1) - f(x_0)| \leq L|x_1 - x_0|$ for all $x_0, x_1 \in \mathbb{R}$.)

Corollary 3 *Assume that f is a Darboux function which satisfies the condition (N), $A \in \mathcal{N}$, $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$ there exists an $i \leq n$ with $f^i(x) \in A$. Then $f(a) = a$ for exactly one $a \in A$.*

$A \subset \mathbb{R}$ is called to be a *strong measure zero set* iff given any sequence $(\varepsilon_n)_n$ of positive numbers, A can be covered by a sequence of sets $(A_n)_n$ with diam$(A_n) < \varepsilon_n$ (where diam(A) denotes the diameter of A) [1]. Let $\mathcal{S}$ denote the class of all strong measure zero sets. Recall that $f(A)$ is strong measure zero whenever f is continuous and $A \in \mathcal{S}$ [6].

Corollary 4 *Assume that f is continuous, $A \in \mathcal{S}$, $n \in \mathbb{N}$ and for every $x \in \mathbb{R}$ there exists an $i \leq n$ such that $f^i(x) \in A$. Then $f(a) = a$ for exactly one $a \in A$.*

Note that there exist Darboux functions f with $f^2(x) = 1$ for all $x \in \mathbb{R}$ which are discontinuous.

Though in Proposition 1 the assumption of boundedness of A cannot be dropped, we have the following:

Proposition 4 *Assume that f is a Darboux function and for each $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $f^n(x) = x$. Then either $f = \mathrm{id}_{\mathbb{R}}$ (where $\mathrm{id}_{\mathbb{R}}$ denotes the identity on $\mathbb{R}$) or f is continuous and decreasing with $f = f^{-1}$.*

PROOF. First note that f is injective. Indeed, suppose $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$. Set $n_1 = \min\{n : f^n(x_1) = x_1\}$ and $n_2 = \min\{n : f^n(x_2) = x_2\}$. If $n_1 = n_2$ then $x_1 = x_2$. So we may assume that $n_1 < n_2$. Let $n_2 = kn_1 + r$, where $k, r \in \mathbb{N}$ and $0 \leq r < n_1$. Then $x_2 = f^{n_2}(x_2) = f^{n_2}(x_1) = f^r\left((f^{n_1})^k(x_1)\right) = f^r(x_1) = f^r(x_2)$, a contradiction. Hence f is continuous (see, e.g., [2]) and monotonic. We consider two casses.

(1.) f is increasing. Fix an $x \in \mathbb{R}$. Then f^n is increasing and (by induction) $f^n(x) > x$ whenever $f(x) > x$ and $f^n(x) < x$ whenever $f(x) < x$ for each $n \in \mathbb{N}$. Hence $f(x) = x$ and $f = \mathrm{id}_{\mathbb{R}}$.

(2.) f is decreasing. Fix an $x \in \mathbb{R}$ and put $x_1 = f(x)$. We shall verify that $f(x_1) = x$. Obviously it is sufficient to consider that $x_1 \neq x$, so either $x < x_1$ or $x_1 < x$.

(2.1.) Assume $x < x_1$ and suppose that $f(x_1) \neq x$, so either $f(x_1) < x$ or $f(x_1) > x$.

(2.1.1.) Suppose $f(x_1) < x$. We can prove by induction that $f^n(x) > x$ when n is odd and $f^n(x) < x$ when n is even. In fact, this is true for $n = 1$ and $n = 2$. Assume that $f^{2n+1}(x) > x$ and $f^{2n+2}(x) < x$. Then $f^{2n+3}(x) > f(x) = x_1 > x$. Moreover, $f^{2n+4}(x) < f(x_1) < x$, which completes the induction. Thus $f^n(x) \neq x$ for all positive integer n, a contradiction.

(2.1.2.) Let $f(x_1) > x$. We verify by induction that $x < f^n(x) < x_1$ for all $n > 1$, which contradicts the assumption on f.

Therefore $f(x_1) = x$ if $x < x_1$.

(2.2.) Now suppose that $x_1 < x$. Since f is decreasing, $f(f(x)) > f(x)$. Then, by (2.1.), $f^3(x) = f(x)$. Since f is injective, $f^2(x) = x$, so $f(x_1) = x$. □

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