Real Analysis Exchange
Vol. 20(1), 1994/95, pp. 336–339

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A NOTE ON MAJOR AND MINOR FUNCTION FOR THE PERRON INTEGRAL

Abstract

A function f that is almost everywhere the derivative of a continuous ACG^* function is Perron integrable; a proof of this is given by a direct construction of continuous major and minor functions for f.

The theorem of Hake-Alexandroff-Looman [1] shows that Perron integral using only continuous major and minor functions and that using general major and minor functions are equivalent. In that Proof, a basic proof, namely, the proof by category argument is used. But, a direct constructive proof of the theorem has been elusive for a long time. In this note, we give a constructive proof, in other words, we provide a technique which directly constructs continuous major and minor functions on the basis of the primitive of a P-integrable function. We remark that we have carried forward the Henstock method [2, p. 194].

A function U is said to be a major function of f on $[a, b]$ if for every $x \in [a, b]$

$$\underline{D}U(x) \geq f(x)$$

where $\underline{D}$ denotes the lower derivative, and $\underline{D}U(x) > -\infty$ for all x. A function V is said to be a minor function of f on $[a, b]$ if $-V$ is a major function of $-f$ on $[a, b]$.

A function f is said to be P-integrable on $[a, b]$ if f has both major and minor functions and

$$-\infty < \inf\{U(b) - U(a)\} = \sup\{V(b) - V(a)\} < +\infty$$

Key Words: Perron integral, major & minor functions, ACG^*, Lusin condition
Mathematical Reviews subject classification: Primary: 26A39
Received by the editors December 21, 1993

where the infimum is over all major functions U of f on $[a,b]$ and the supremum over all minor functions V of f on $[a,b]$. The common value of $\inf\{U(b)-U(a)\}$ and $\sup\{V(b)-V(a)\}$ is defined to be the P-integral of f on $[a,b]$.

A function f is said to be P_0-integrable on $[a,b]$, if

$$\inf\{U_0(b)-U_0(a)\} = \sup\{V_0(b)-V_0(a)\}$$

where the infimum is over all continuous major functions U_0 of f and the supremum over all continuous minor functions V_0 of f. The common value is the P_0-integral of f on $[a,b]$.

A function F is ACG^* if $[a,b]=\cup_{i=1}^{\infty}X_i$ and F is $AC^*(X_i)$ for each i, i.e. for every $\varepsilon>0$ there is $\eta>0$ such that for every finite or infinite sequence of nonoverlapping intervals $\{[a_k,b_k]\}$ with at least one of a_k, b_k belonging to X_i for all k and satisfying

$$\sum_k |b_k-a_k|<\eta \quad \text{we have} \quad \sum_k |F(b_k)-F(a_k)|<\varepsilon.$$

It has been shown [3,4] that the above definition is equivalent to the classical definition as given in Saks [1]. In particular, if F is the primitive of P-integrable function f on $[a,b]$, then F is ACG^*.

Theorem 1 *If f is P_0-integrable on $[a,b]$, then f is P-integrable on $[a,b]$.*

This is obvious. Conversely, we have the following:

Theorem 2 *If f is P-integrable on $[a,b]$ with the primitive F, ε is a given positive number, then we can directly construct a continuous major function U_0 and a continuous minor function V_0 such that*

$$U_0(a)=V_0(a)=0,\ -\infty\neq \underline{D}U_0(x)\geq f(x)\geq \bar{D}V_0(x)\neq+\infty \text{ for all } x\in[a,b].$$

and $U_0(b)-V_0(b)<\varepsilon$. That is, f is P_0-integrable.

PROOF. If f is P-integrable on $[a,b]$, it is obvious, that the primitive F of f is continuous satisfying $\underline{D}F(x)=F'(x)=f(x)=\bar{D}F(x)$ almost everywhere, and $F'(x)=\pm\infty$ for x belonging to a set of measure zero only.

Let Z be the set of points x of $[a,b]$ at which $\underline{D}F(x)=-\infty$, and the measure $|Z|=0$. Since F is ACG^*, there are $X_1,X_2,\ldots$, with union $[a,b]$, such that F is $AC^*(X_i)$ for each i. Let $Y_i=Z\cap X_i$ for $i=1,2,\ldots$. Since F is $AC^*(Y_i)$, for every $\varepsilon>0$ there is $\eta_i>0$ such that for any sequence of non-overlapping intervals $\{I_j\}$ with $I_j\cap Y_i\neq\phi$ for each j and satisfying

$$\sum_j |I_j|<\eta_i \quad \text{we have} \quad \sum_j |F(I_j)|<\varepsilon 2^{-i-2}.$$

Here $F(I)$ denotes $F(v) - F(u)$, where $I = [u, v]$. Choose an open set G_i (which is the union of a sequence of open intervals) such that

$$|G_i| < \eta_i \text{ and } G_i \supset Y_i,$$

where $|G_i|$ denotes the measure of G_i.

Now define

$$H_i(x) = \sup \sum_j |F(I_j)|$$

where the supremum is taken over all sequences of non-overlapping intervals $\{I_j\}$ each of which is contained in $G_i \cap [a, x]$ with $I_j \cap Y_i \neq \phi$ for each j. Obviously, H_i is continuous, nondecreasing

$$H_i(a) = 0,\ H_i(b) \leq \varepsilon 2^{-i-2}, \text{ and } H_i(y) - H_i(x) \geq |F(y) - F(x)|$$

whenever $[x, y] \subset G_i$ and $[x, y] \cap Y_i \neq \phi$. Put

$$H(x) = \sum_{i=1}^{\infty} H_i(x) \text{ for } x \in [a, b].$$

Then H is still increasing on $[a, b]$, $H(a) = 0$, and $H(b) \leq \varepsilon/4$. Therefore for every $x \in Z$ we have

$$\underline{D}(F(x) + H(x)) \geq 0.$$

Hence we have removed the points x at which $\underline{D}U(x) = -\infty$.

Again, let $Z_1 = \{x | x \in [a, b],\ f(x) > \underline{D}(F(x) + H(x)) > -\infty\}$. It is obvious that Z_1 is a set of measure zero. It follows from [2, p. 43] that for given $\varepsilon > 0$, there is a continuous and monotone increasing function W such that

$$W(a) = 0,\ W(b) < \varepsilon/4, \text{ and } W'(x) = +\infty \text{ for all } x \text{ in } Z_1.$$

Since W is monotone and increasing we have clearly $\underline{D}W(x) \geq 0$.

Put $U_0(x) = F(x) + H(x) + W(x)$, we obtain

$$\begin{aligned} \underline{D}U_0(x) &= \underline{D}(F(x) + H(x) + W(x)) \geq \underline{D}(F(x) + H(x)) + \underline{D}(W(x)) \\ &= +\infty \text{ for all } x \in Z_1. \end{aligned}$$

Therefore we have

$$\underline{D}(U_0(X)) \geq f(x) \text{ for every } x \in [a, b], \text{ and } U_0(b) < F(b) + \varepsilon/2.$$

Similarly, we can directly construct a continuous major function V_0, such that

$$V_0(a) = 0,\ +\infty \neq \bar{D}V_0(x) \leq f(x) \text{ for all } x \in [a, b], \text{ and } V_0(b) > F(b) - \varepsilon/2.$$

Hence f is P_0-integrable on $[a, b]$. The proof is complete. □

We remark that, as can be seen from the proof above, the full condition of ACG^* is not used. Indeed, it is sufficient to assume F to have the so-called strong Lusin condition [4] in the place of ACG^*.

References

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