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CARDINAL INVARIANTS CONCERNING FUNCTIONS WHOSE PRODUCT IS ALMOST CONTINUOUS

Abstract

We prove that the smallest cardinality of a family $\mathcal{F}$ of real functions for which there is no non-zero function $g : \mathbb{R} \to \mathbb{R}$ with the property that $f \cdot g$ is almost continuous (connected, Darboux function, respectively) for all $f \in \mathcal{F}$, is equal to the cofinality of the continuum.

We shall consider real functions defined on a real interval. No distinction is made between a function and its graph. The notation $[f > 0]$ means the set $\{x : f(x) > 0\}$. Likewise for $[f = 0]$, $[f \neq 0]$, etc. If A is a planar set, we denote its x-projection by $\mathrm{dom}(A)$ and y-projection by $\mathrm{rng}(A)$. We say that a set $A \subset \mathbb{R}$ is bilaterally c-dense at a point $x \in \mathbb{R}$ if $\mathrm{card}(A \cap [x, x + \varepsilon)) = 2^\omega$ and $\mathrm{card}(A \cap (x - \varepsilon, x]) = 2^\omega$ for each $\varepsilon > 0$.

A function f is said to be Darboux if $f(C)$ is connected whenever C is a connected subset of the domain of f. If each open set containing f also contains a continuous function with the same domain as f, then f is almost continuous [7]. It is well-known that if $f : I \longrightarrow \mathbb{R}$ is almost continuous, then f is connected and, therefore, it possesses the Darboux property [7]. Moreover, if f intersects all closed subsets K of $\mathbb{R}^2$ with $\mathrm{dom}(K)$ being a non-degenerate interval and $\mathrm{rng}(K) = \mathbb{R}$, then f is almost continuous [2]. In this paper every such set is called a blocking set. The family of all almost continuous functions will be denoted by $\mathcal{AC}$, the family of all connected functions will be denoted by $\mathcal{C}onn$ and the family of all Darboux functions by $\mathcal{D}$.

For arbitrary families $\mathcal{F}_0$ and $\mathcal{F}_1$ of real functions let us define the following conditions:

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$U_m(\mathcal{F}_0;\mathcal{F}_1)$: there exists a non-zero function g such that $f\cdot g\in\mathcal{F}_1$ whenever $f\in\mathcal{F}_0$.

$U_m^*(\mathcal{F}_0;\mathcal{F}_1)$: there exists a non-zero function $g\in\mathcal{F}_1$ such that $f\cdot g\in\mathcal{F}_1$ whenever $f\in\mathcal{F}_0$.

Let $m(\mathcal{F}_1)$ denote the least cardinal κ for which there exists a family $\mathcal{F}_0$ of real functions such that $\mathrm{card}(\mathcal{F}_0)=\kappa$ and $U_m(\mathcal{F}_0;\mathcal{F}_1)$ is false. We put $m(\mathcal{F}_1)=0$ if $U_m(\mathbb{R}^{\mathbb{R}};\mathcal{F}_1)$ holds. Similarly we define the cardinal $m^*(\mathcal{F}_1)$.

Note that $U_m^*(\mathcal{F}_0;\mathcal{F}_1)\Rightarrow U_m(\mathcal{F}_0;\mathcal{F}_1)$ for any families $\mathcal{F}_0$, $\mathcal{F}_1$. Moreover $U_m^*(\mathcal{F}_0;\mathcal{F}_1)\equiv U_m(\mathcal{F}_0;\mathcal{F}_1)$ whenever $f\equiv 1$ belongs to $\mathcal{F}_0$. Hence $m^*(\mathcal{F}_1)\leq m(\mathcal{F}_1)$ for every family $\mathcal{F}_1$ and $m^*(\mathcal{F}_1)=m(\mathcal{F}_1)$ if $m(\mathcal{F}_1)$ is infinity.

The problem to determine how big can be the cardinal $m(\mathcal{AC})$ was considered in [4]. (See also [5].) Since $U_m(\mathcal{F};\mathcal{AC})$ is false for the family $\mathcal{F}$ of all singletons, $m(\mathcal{AC})\leq 2^\omega$. (See [3].) Assuming that the additivity of the ideal of all sets of the first category is 2^ω (which is a consequence of Martin's Axiom and therefore also of the Continuum Hypothesis [6]) it is proved in [3] that $m(\mathcal{AC})=2^\omega$. This suggests the following question (cf. [4, Problem 6.2] and [5, Problem 1.7.2, p. 84]):

Problem 1 *Can the equality $m(\mathcal{AC})=2^\omega$ be proved in ZFC?*

In the present note we answer this problem in the negative by showing that $m(\mathcal{AC})=\mathrm{cf}(2^\omega)$, where $\mathrm{cf}(2^\omega)$ denotes, as usually, the cofinality of the continuum.[1]

Theorem 1 *For every family $\mathcal{F}$ of real functions with* $\mathrm{card}(\mathcal{F})\leq 2^\omega$ *the following conditions are equivalent:*

1 $U_m^*(\mathcal{F};\mathcal{AC})$;

2 $U_m^*(\mathcal{F};Conn)$;

3 $U_m^*(\mathcal{F};\mathcal{D})$;

4 there exists a non-empty bilaterally c-dense in itself set $A\subset\mathbb{R}$ such that $A\cap[f\neq 0]$ is bilaterally c-dense in itself for each $f\in\mathcal{F}$.

PROOF. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Assume that g is a non-zero Darboux function and $f\cdot g$ are Darboux for each $f\in\mathcal{F}$. Put $A=[g\neq 0]$. By the intermediate value property of g, A is bilaterally c-dense in itself. Now fix $f\in\mathcal{F}$ and $x\in A\cap[f\neq 0]$.

[1]Note that the analogous result concerning the addition is independent of ZFC. This result was proved by A. Miller during the Joint US-Polish Workshop in Real Analysis, Łódź, Poland, July 14–19, 1994, cf. [1].

Then $(f \cdot g)(x) \neq 0$ and, since $f \cdot g$ is Darboux, $[f \cdot g \neq 0]$ is bilaterally c-dense at x. Therefore $A \cap [f \neq 0]$ has the same property.

(4) $\Rightarrow$ (1) Arrange all blocking sets K with $A \cap \operatorname{dom}(K) \neq \emptyset$ and all horizontal lines in a sequence $(K_\alpha)_{\alpha<2^\omega}$, and all functions $f \in \mathcal{F}$ in a sequence $(f_\alpha)_{\alpha<2^\omega}$. Note that $\operatorname{card}(A \cap \operatorname{dom}(K_\alpha)) = 2^\omega$ for every $\alpha < 2^\omega$. Moreover, we can assume that $f \equiv 1$ belongs to $\mathcal{F}$. Let $\varphi : 2^\omega \to 2^\omega \times 2^\omega$ be a bijection and $\varphi = (\varphi_0, \varphi_1)$. For every $\gamma < 2^\omega$ choose (x_γ, y_γ) in the following way. Fix $\gamma < 2^\omega$. Let $\varphi(\gamma) = (\alpha, \beta)$. We cosider two cases.

1. If $\operatorname{card}(A \cap \operatorname{dom}(K_\beta) \cap [f_\alpha \neq 0]) = 2^\omega$ then $(x_\gamma, y_\gamma) \in K_\beta$, $x_\gamma \neq 0$, $x_\gamma \notin \{x_\xi,\ \xi < \gamma\}$ and $f_\alpha(x_\gamma) \neq 0$.
2. If $\operatorname{card}(A \cap \operatorname{dom}(K_\beta) \cap [f_\alpha \neq 0]) < 2^\omega$ then $(x_\gamma, y_\gamma) = (0,0)$.

Now define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} y_\gamma / f_\alpha(x_\gamma) & \text{if } x = x_\gamma,\ \alpha = \varphi_0(\gamma) \text{ , and } \gamma 2^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

We shall verify that $(f \cdot g) \cap K \neq \emptyset$ for every $f \in \mathcal{F}$ and each blocking set K.

If $\operatorname{dom}(K) \cap A = \emptyset$ then $g|\operatorname{dom}(K) \equiv 0$, so $(f \cdot g)|\operatorname{dom}(K) \equiv 0$. Since $\operatorname{rng}(K) = \mathbb{R}$, $(f \cdot g) \cap K \neq \emptyset$. Similarly, if $\operatorname{dom}(K) \cap A \subset [f = 0]$ then $\operatorname{dom}(K) \subset [f \cdot g = 0]$, so $(f \cdot g) \cap K \neq \emptyset$.

If $\operatorname{dom}(K) \cap A \cap [f \neq 0] \neq \emptyset$ then $\operatorname{card}(A \cap \operatorname{dom}(K) \cap [f \neq 0]) = 2^\omega$ and there exist $\alpha, \beta < 2^\omega$ such that $f = f_\alpha$ and $K = K_\beta$. Then $(x_\gamma, y_\gamma) \in (f_\alpha \cdot g) \cap K_\beta$ for $\gamma = \varphi^{-1}(\alpha, \beta)$.

Since $1 \in \mathcal{F}$, g is almost continuous. Since g meets every horizontal line, $\operatorname{rng}(g) = \mathbb{R}$ and hence $g \not\equiv 0$.

Note moreover that $[g \neq 0] \subset A$. □

Corollary 1 $m(\mathcal{AC}) = m(\mathcal{C}onn) = m(\mathcal{D}) = \operatorname{cf}(2^\omega)$

PROOF. Assume that $\operatorname{card}(\mathcal{F}) < \operatorname{cf}(2^\omega)$. Set

$$A_f = \{x :\ f(x) \neq 0 \text{ and } [f \neq 0] \text{ is not bilaterally } c\text{-dense at } x\}\,.$$

Note that $\operatorname{card}(A_f) < 2^\omega$ for all $f \in \mathcal{F}$. (Indeed, for every $B \subset \mathbb{R}$ the set of all $x \in B$ such that B is not bilaterally c-dense at x can be represented as the union $B^0 \cup B^- \cup B^+$, where B^0 denotes the set of all $x \in B$ for which there exist rationals p, q such that $p < x < q$ and $\operatorname{card}((p,q) \cap B) < 2^\omega$; B^+ is the set of all $x \in B \setminus B^0$ for which there exists a q such that $x < q$ and $(x,q) \cap B \subset B^0$ and B^- is the set of all $x \in B \setminus B^0$ for which there exists a p

such that $p < x$ and $(p, x) \cap B \subset B^0$. It is easy to check that $\mathrm{card}(B^0) < 2^\omega$, $\mathrm{card}(B^-) \leq \omega$ and $\mathrm{card}(B^+) \leq \omega$ for all $B \subset \mathbb{R}$.) Hence $\mathrm{card}(\bigcup_{f \in \mathcal{F}} A_f) < 2^\omega$ and the condition (4) from Theorem 1 is fulfilled by $A = \mathbb{R} \setminus \bigcup_{f \in \mathcal{F}} A_f$. By that Theorem, $m^*(\mathcal{AC}) \geq \mathrm{cf}(2^\omega)$, so $m(\mathcal{AC}) \geq \mathrm{cf}(2^\omega)$.

Now let $\{A_\alpha : \alpha < \mathrm{cf}(2^\omega)\}$ be a family of subsets of $\mathbb{R}$ with $\mathbb{R} = \bigcup_{\alpha < \mathrm{cf}(2^\omega)} A_\alpha$ and $\mathrm{card}(A_\alpha) < 2^\omega$ for each $\alpha < \mathrm{cf}(2^\omega)$. For every $\alpha < \mathrm{cf}(2^\omega)$ let f_α be a characteristic function of A_α, i.e.,

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x \in A_\alpha, \\ 0 & \text{if } x \notin A_\alpha. \end{cases}$$

Assume that $g \cdot f_\alpha$ is Darboux for every $\alpha < \mathrm{cf}(2^\omega)$. Then $\mathrm{rng}(g \cdot f_\alpha)$ is an interval, so $f_\alpha \cdot g \equiv 0$ and consequently, $g(x) = 0$ for every $x \in A_\alpha$. Hence $g = 0$ and $m(\mathcal{D}) \leq \mathrm{cf}(2^\omega)$. Because $m(\mathcal{AC}) \leq m(\mathcal{C}onn) \leq m(\mathcal{D})$, we have the desired equalities. □

The corollary above can be improved in the following way.

Theorem 2 *Let $\mathcal{B} \subset P(\mathbb{R})$ be a σ-algebra containing a hereditarily measurable set of the size 2^ω and $\mathcal{M}(\mathcal{B})$ be the class of all $\mathcal{B}$-measurable real functions. Then $m(\mathcal{AC} \cap \mathcal{M}(\mathcal{B})) = m(\mathcal{C}onn \cap \mathcal{M}(\mathcal{B})) = m(\mathcal{D} \cap \mathcal{M}(\mathcal{B})) = \mathrm{cf}(2^\omega)$.*

PROOF. Obviously, we have to prove only one equality: $m(\mathcal{AC} \cap \mathcal{M}(\mathcal{B})) \geq \mathrm{cf}(2^\omega)$. Let X be a hereditarily $\mathcal{B}$ measurable set with $\mathrm{card}(X) = 2^\omega$. We can assume that X is bilaterally c-dense in itself. Let $\mathcal{F}$ be a family of functions of the size less than $\mathrm{cf}(2^\omega)$. For each $f \in \mathcal{F}$ let A_f be defined as above. Then $\mathrm{card}(\bigcup_{f \in \mathcal{F}} A_f) < 2^\omega$ and $A = X \setminus \bigcup_{f \in \mathcal{F}} A_f$ satisfies the condition (4) of Theorem 1, so $U_m(\mathcal{F}, \mathcal{AC})$ is fulfilled by the function g such that $[g \neq 0] \subset A$. Hence g is $\mathcal{B}$-measurable. □

Corollary 2 *Let $\mathcal{L}$ denote the family of all Lebesgue measurable functions. Then $m(\mathcal{AC} \cap \mathcal{L}) = m(\mathcal{C}onn \cap \mathcal{L}) = m(\mathcal{D} \cap \mathcal{L}) = \mathrm{cf}(2^\omega)$.*

Corollary 3 *Let $\mathcal{K}$ denote the family of all functions with the Baire propery. Then $m(\mathcal{AC} \cap \mathcal{K}) = m(\mathcal{C}onn \cap \mathcal{K}) = m(\mathcal{D} \cap \mathcal{K}) = \mathrm{cf}(2^\omega)$.*

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